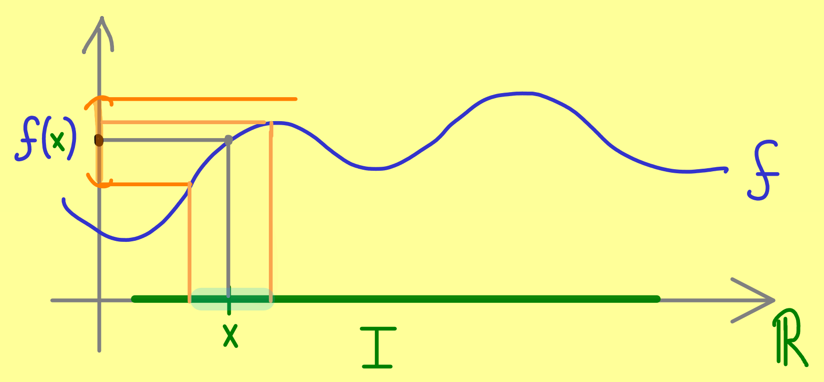
The Bright Side of Mathematics

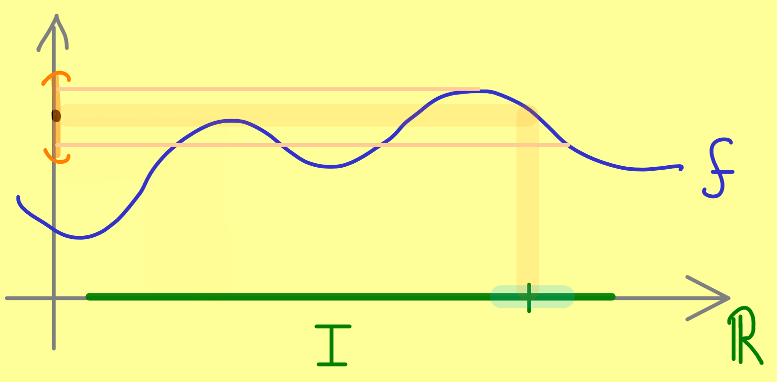
Absolutely Continuous Functions - Part 1



$f: I \longrightarrow \mathbb{R}$

Definition: $f: I \longrightarrow \mathbb{R}$ is called <u>continuous</u> if

$$\forall x \in I \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall \tilde{x} \in I: \; |x - \tilde{x}| < \delta \implies |f(x) - f(\tilde{x})| < \varepsilon$$



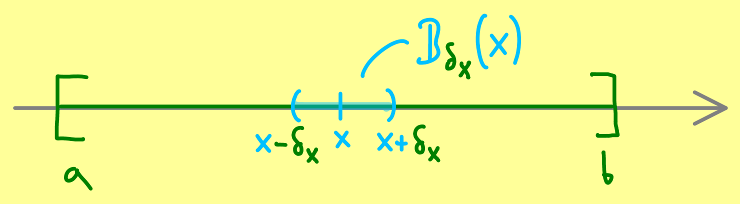
Definition: $f: I \longrightarrow \mathbb{R}$ is called <u>uniformly continuous</u> if

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x, \tilde{x} \in I: \; |x - \tilde{x}| < \delta \implies |f(x) - f(\tilde{x})| < \varepsilon$$

Proposition: $f: [a,b] \longrightarrow \mathbb{R}$: f is continuous $\iff$ f is uniformly continuous

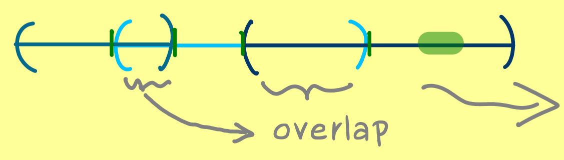
Proof: $(\Rightarrow)$ For given $x \in [a,b]$ and $\varepsilon > 0$, there is $\delta_x > 0$ with the property:

$$\forall \tilde{x} \in [a,b]: \; |x - \tilde{x}| < \delta_x \implies |f(x) - f(\tilde{x})| < \varepsilon$$



$[a,b] \subseteq \bigcup_{x \in [a,b]} B_{\delta_x}(x)$ open covering

$\overset{\text{compactness}}{\implies} \; [a,b] \subseteq \bigcup_{j=1}^{m} B_{\delta_{x_j}}(x_j)$



$\square$

Definition: $f: [a,b] \longrightarrow \mathbb{R}$ is called <u>absolutely continuous</u> if

$\forall \varepsilon > 0 \;\; \exists \delta > 0$: for any finite collection of pairwise disjoint intervals $(x_k, \tilde{x}_k) \subseteq [a,b]$ $\quad (k = 1, \dots, m)$

we have: $\sum_{k=1}^{m} |x_k - \tilde{x}_k| < \delta \implies \sum_{k=1}^{m} |f(x_k) - f(\tilde{x}_k)| < \varepsilon$