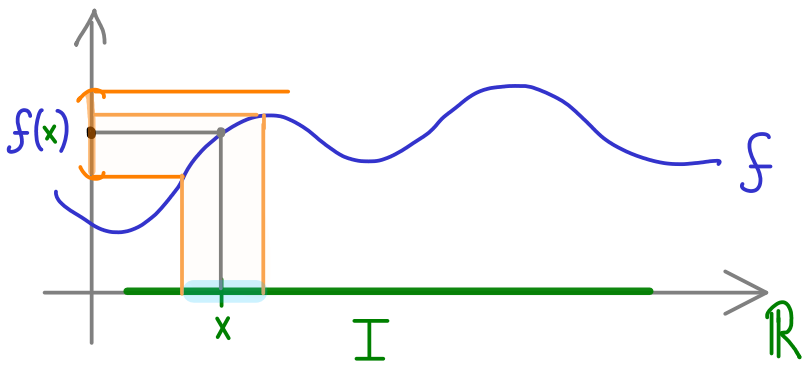
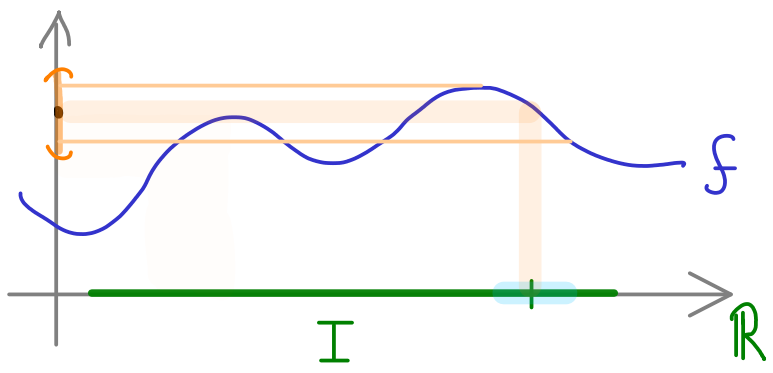
Absolutely Continuous Functions - Part 1



$f: I \longrightarrow \mathbb{R}$

Definition: $f: I \longrightarrow \mathbb{R}$ is called continuous if

$$\forall x \in I \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \tilde{x} \in I: \quad |x - \tilde{x}| < \delta \implies |f(x) - f(\tilde{x})| < \varepsilon$$



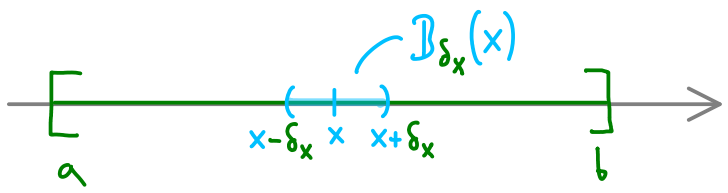
Definition: $f: I \longrightarrow \mathbb{R}$ is called uniformly continuous if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, \tilde{x} \in I: \quad |x - \tilde{x}| < \delta \implies |f(x) - f(\tilde{x})| < \varepsilon$$

Proposition: $f: [a,b] \longrightarrow \mathbb{R}$: f is continuous $\iff$ f is uniformly continuous

Proof: ($\Longrightarrow$) For given $x \in [a,b]$ and $\varepsilon > 0$, there is $\delta_x > 0$ with the property:

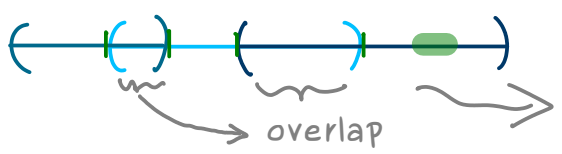
$$\forall \tilde{x} \in [a,b]: \quad |x - \tilde{x}| < \delta_x \implies |f(x) - f(\tilde{x})| < \varepsilon$$



$$[a,b] \subseteq \bigcup_{x \in [a,b]} B_{\delta_x}(x) \quad \text{open covering}$$

compactness

$$\implies [a,b] \subseteq \bigcup_{j=1}^{m} B_{\delta_{x_j}}(x_j)$$



□

Definition: $f: [a,b] \longrightarrow \mathbb{R}$ is called absolutely continuous if

$\forall \varepsilon > 0 \quad \exists \delta > 0$: for any finite collection of pairwise disjoint intervals

$$(x_k, \tilde{x}_k) \subseteq [a,b] \qquad (k = 1, \dots, m)$$

we have: $\sum_{k=1}^{m} |x_k - \tilde{x}_k| < \delta \implies \sum_{k=1}^{m} |f(x_k) - f(\tilde{x}_k)| < \varepsilon$