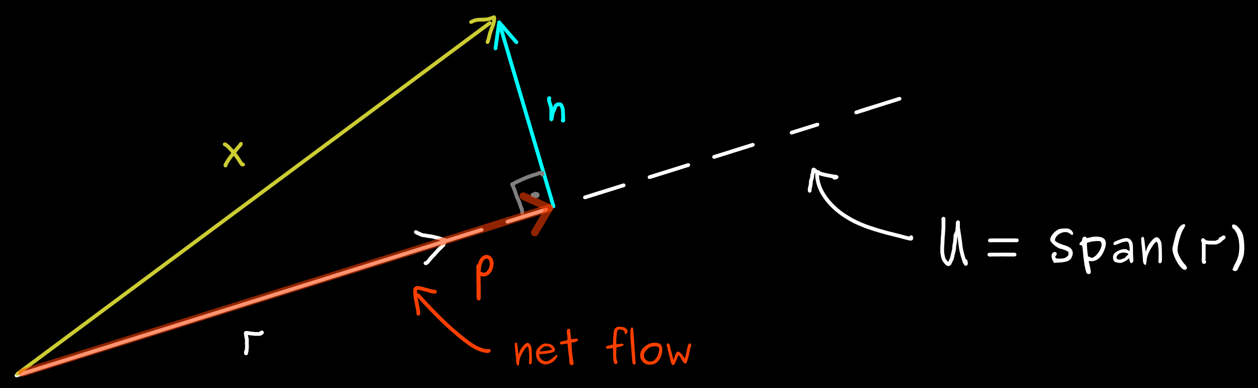
Abstract Linear Algebra - Part 14



Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

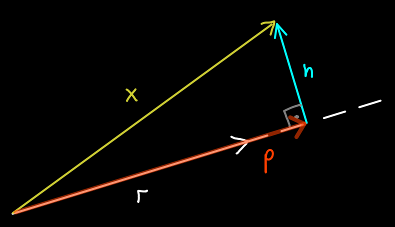
Let $\mathcal{U} \subseteq V$ be a subspace with $\mathcal{U} = \mathrm{Span}(r)$, $r \neq 0$.

For $x \in V$ and a decomposition $x = p + n$ with $p \in \mathcal{U}$, $n \perp r$,

we call:

p	orthogonal projection of x onto $\mathcal{U}$
n	normal component of x with respect to $\mathcal{U}$

Let's show the uniqueness: Assume $x = \underbrace{p}_{\in \mathcal{U}} + \underbrace{n}_{\in \mathcal{U}^\perp}$, $x = \underbrace{\tilde{p}}_{\in \mathcal{U}} + \tilde{n}_{\in \mathcal{U}^\perp}$



$$\Longrightarrow p + n = \tilde{p} + \tilde{n} \Longrightarrow \underbrace{p - \tilde{p}}_{\in \mathcal{U}} = \underbrace{\tilde{n} - n}_{\in \mathcal{U}^\perp}$$

$$\Longrightarrow 0 = \langle p - \tilde{p}, \tilde{n} - n \rangle = \begin{cases} \langle p - \tilde{p}, p - \tilde{p} \rangle \\ \langle \tilde{n} - n, \tilde{n} - n \rangle \end{cases}$$

inner product is positive definite

$$\Longrightarrow p - \tilde{p} = 0 = \tilde{n} - n \Longrightarrow p = \tilde{p} \quad \text{and} \quad n = \tilde{n}$$

Existence: $p \in U = \mathrm{Span}(r) \implies p = \lambda \cdot r$ for $\lambda \in \mathbb{F}$

$$\langle r, x\rangle = \langle r, \lambda\cdot r + n\rangle = \lambda \langle r, r\rangle + \underbrace{\langle r, n\rangle}_{=0}$$

$$\implies \lambda = \frac{\langle r, x\rangle}{\langle r, r\rangle} \rightsquigarrow p = \frac{\langle r, x\rangle}{\langle r, r\rangle}\cdot r \quad , \; n = x - p \; \checkmark$$