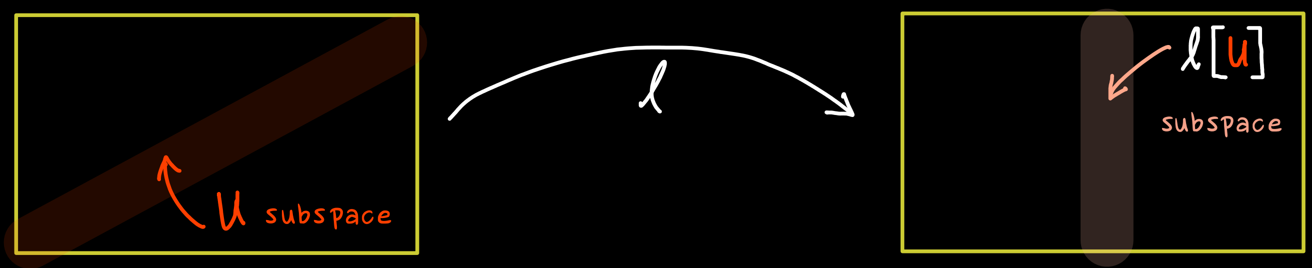
Abstract Linear Algebra - Part 38

Definition: Let V be an $\mathbb{F}$-vector space and $\ell : V \longrightarrow V$ be a linear map.



A subspace U is called invariant under ℓ if $\ell[U] \subseteq U$.

Consequence: $\ell|_U : U \longrightarrow U$ linear map

Important application: $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ eigenvalue of A.

We have generalized eigenspaces:

$$\{0\} \subsetneq \mathrm{Ker}\big((A - \lambda \cdot \mathbb{1})^1\big) \subsetneq \mathrm{Ker}\big((A - \lambda \cdot \mathbb{1})^2\big) \subsetneq \cdots \subsetneq \mathrm{Ker}\big((A - \lambda \cdot \mathbb{1})^{\overset{\text{Fitting index}}{d}}\big)$$

$$1 \leq d \leq n$$

$$\mathbb{C}^n \supsetneq \mathrm{Ran}\big((A - \lambda \cdot \mathbb{1})^1\big) \supsetneq \mathrm{Ran}\big((A - \lambda \cdot \mathbb{1})^2\big) \supsetneq \cdots \supsetneq \mathrm{Ran}\big((A - \lambda \cdot \mathbb{1})^d\big)$$

Then: $\mathrm{Ker}\big((A - \lambda \cdot \mathbb{1})^d\big)$ and $\mathrm{Ran}\big((A - \lambda \cdot \mathbb{1})^d\big)$ are invariant under A

Proof: $(A - \lambda \cdot \mathbb{1})x = Ax - \lambda x$. So: $\begin{matrix} x \in U \\ (A - \lambda \cdot \mathbb{1})x \in U \end{matrix} \Longrightarrow Ax = \underbrace{(A - \lambda \cdot \mathbb{1})x}_{\in U} + \underbrace{\lambda x}_{\in U}$

Define: $N := A - \lambda \cdot \mathbb{1}$

(1) Show $\mathrm{Ker}(N^d)$ is invariant: $x \in \mathrm{Ker}(N^d) \Longrightarrow N^d(Nx) = N \underbrace{N^d x}_{=0} = 0$

$$\Longrightarrow Nx \in \mathrm{Ker}(N^d)$$

(2) Show $\mathrm{Ran}(N^d)$ is invariant: $y \in \mathrm{Ran}(N^d) \Longrightarrow$ there is $x \in \mathbb{C}^n$ with $y = N^d x$

$$\Longrightarrow Ny = N(N^d x) = N^d \underbrace{(Nx)}_{= \tilde{x} \in \mathbb{C}^n} \in \mathrm{Ran}(N^d) \qquad \square$$