The Bright Side of
Mathematics

Abstract Linear Algebra - Part 28

Fact: $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$ are different but

they describe the same linear map $\ell : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\ell(p) = p'$

with respect to different bases.

Question: $\ell : V \longrightarrow W$ linear , $A = \ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}$.

For another $\tilde{A} \in \mathbb{F}^{m \times n}$, can we find bases such that $\tilde{A} = \ell_{\tilde{\mathcal{C}} \leftarrow \tilde{\mathcal{B}}}$?

If YES!, then we say A and $\tilde{A}$ are equivalent.

Picture:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\mathcal{B}}\ \underset{\text{(standard basis)}}{\mathbb{F}^n} \arrow[r, "\ell_{\mathcal{C} \leftarrow \mathcal{B}}"] \arrow[d, shift left, "T_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}}"] & \underset{\text{(standard basis)}}{\mathbb{F}^m}\ \boxed{\mathcal{C}} \arrow[d, shift left, "T_{\tilde{\mathcal{C}} \leftarrow \mathcal{C}}"] \\
\boxed{\tilde{\mathcal{B}}}\ \underset{\text{(standard basis)}}{\mathbb{F}^n} \arrow[r, "\ell_{\tilde{\mathcal{C}} \leftarrow \tilde{\mathcal{B}}}"] \arrow[u, shift left, "T_{\mathcal{B} \leftarrow \tilde{\mathcal{B}}}"] & \underset{\text{(standard basis)}}{\mathbb{F}^m}\ \boxed{\tilde{\mathcal{C}}} \arrow[u, shift left, "T_{\mathcal{C} \leftarrow \tilde{\mathcal{C}}}"]
\end{tikzcd}$$

invertible! change of basis

change of basis invertible!

Definition: A matrix $\tilde{A} \in \mathbb{F}^{m \times n}$ is called equivalent to a matrix $A \in \mathbb{F}^{m \times n}$

if there are invertible matrices $S \in \mathbb{F}^{m \times m}$, $T \in \mathbb{F}^{n \times n}$, such that:

$$\tilde{A} = S\,A\,T\,.$$

We write: $\tilde{A} \sim A$

Remark: $\sim$ defines an equivalence relation on $\mathbb{F}^{m \times n}$:

(1) reflexive: $A \sim A$ for all $A \in \mathbb{F}^{m \times n}$

(2) symmetric: $A \sim B \implies B \sim A$ for all $A, B \in \mathbb{F}^{m \times n}$

(3) transitive: $A \sim B \ \wedge\ B \sim C \implies A \sim C$ for all $A, B, C \in \mathbb{F}^{m \times n}$