

## **The Bright Side of Mathematics**

The following pages cover the whole Advent of Mathematical Symbols course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: <https://tbsom.de/support>

Have fun learning mathematics!

Day 01

# The Bright Side of Mathematics



## Advent of Mathematical Symbols

Kronecker delta:  $\delta_{ij} := \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}$

Example:  $\delta_{12} = 0$  ,  $\delta_{55} = 1$  ,  $\sum_{i,j=1}^5 \delta_{ij} = 5$

Day 02

The Bright Side of  
 Mathematics



# Advent of Mathematical Symbols

Levi-Civita symbol:  $\epsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1, & (i,j,k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0, & \text{else} \end{cases}$

Example:  $(a \times b)_i = \sum_{j,k=1}^3 \epsilon_{ijk} a_j b_k$

Day 03

# The Bright Side of Mathematics



## Advent of Mathematical Symbols

Nabla symbol:  $\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$  or  $\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$

Example:  $f(x_1, x_2) = x_1^3$ ,  $\nabla f(x_1, x_2) = \begin{pmatrix} 3x_1^2 \\ 0 \end{pmatrix}$



# Advent of Mathematical Symbols

Factorial:  $n! := n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$

Example:  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$  ,  $1! = 1$

Recursive definition:  $0! := 1$  ,  $n! := n \cdot (n-1)! \quad (n \in \mathbb{N})$



## Advent of Mathematical Symbols

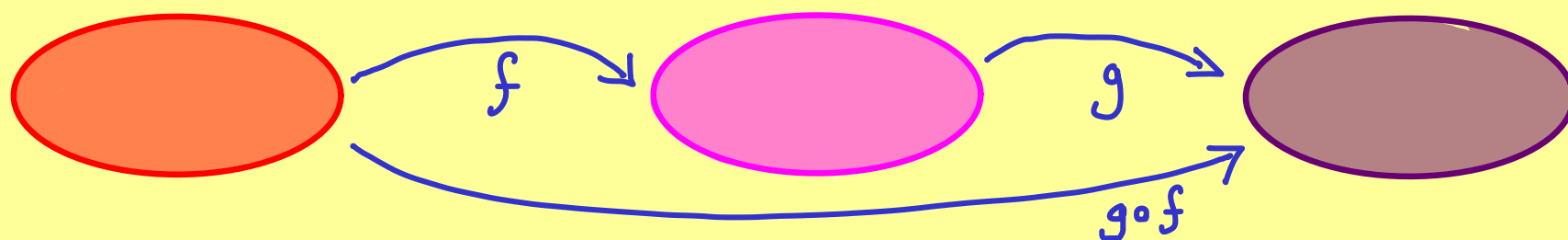
Gamma function:  $\Gamma(z) := \int_0^{\infty} x^{z-1} \cdot e^{-x} dx$ ,  $\text{Re}(z) > 0$

Property:  $\Gamma(n) = (n-1)!$ ,  $\Gamma(z+1) = z \cdot \Gamma(z)$   
for  $n \in \mathbb{N}$



## Advent of Mathematical Symbols

Composition:  $(g \circ f)(x) := g(f(x))$





## Advent of Mathematical Symbols

sum:

$$\sum_{k=1}^n a_k := a_1 + a_2 + \dots + a_n$$

recursive definition:

$$\sum_{k=1}^0 a_k := 0 \quad , \quad \sum_{k=1}^n a_k := \left( \sum_{k=1}^{n-1} a_k \right) + a_n$$





## Advent of Mathematical Symbols

product:  $\prod_{k=1}^n a_k := a_1 \cdot a_2 \cdot \dots \cdot a_n$

recursive definition:  $\prod_{k=1}^0 a_k := 1$  ,  $\prod_{k=1}^n a_k := \left( \prod_{k=1}^{n-1} a_k \right) \cdot a_n$



## Advent of Mathematical Symbols

restriction:  $f|_A : A \rightarrow Y$

For  $f: X \rightarrow Y$  and  $A \subseteq X$ :  $f|_A(x) \stackrel{\text{for all } x \in A}{=} f(x)$



## Advent of Mathematical Symbols

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have:

$$\sigma_k^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_j \sigma_k - \sigma_k \sigma_j = 2i \varepsilon_{jkl} \sigma_l$$



## Advent of Mathematical Symbols

set brackets:  $\{ f(x) \mid x \in A \}$

Example:  $\{ 2 \cdot x + 1 \mid x \in \{0, 1, 2, 3\} \} = \{1, 3, 5, 7\}$



## Advent of Mathematical Symbols

Big O:  $f(x) = \mathcal{O}(g(x)) \quad (x \rightarrow a)$

means:  $|f(x)| \leq M \cdot |g(x)|$

$\left( \limsup_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| < \infty \right)$

Example:  $x^2 + x + 2 = \mathcal{O}(x^2) \quad (x \rightarrow \infty)$

$x^2 + x + 2 = \mathcal{O}(x^3) \quad (x \rightarrow \infty)$



## Advent of Mathematical Symbols

Binomial coefficient: 
$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

① ② ③ ④ ⑤ ⑥ ⑦  
 Take:  $k=3$     ② ③ ⑥  
 or    ⑤ ⑥ ⑦ ...

$n=7$

$$\frac{n \cdot (n-1) \cdot (n-2)}{3 \cdot 2 \cdot 1}$$



## Advent of Mathematical Symbols

Modulo:  $x \bmod n := r \in [0, n)$

with  $x = n \cdot q + r$   
↖ integer

Examples:  $5 \bmod 3 = 2$

$6 \bmod 3 = 0$

$7.1 \bmod 3 = 1.1$

$9.7 \bmod 2.1 = 1.3$

$\begin{array}{l} \downarrow -2.1 \\ 7.6 \\ \downarrow -2.1 \\ 5.5 \rightarrow 3.4 \rightarrow 1.3 \end{array}$



## Advent of Mathematical Symbols

Beta function:  $B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt$

$$B(x, y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$$

$$x, y \in \mathbb{C},$$

$$\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$$





## Advent of Mathematical Symbols

$$f: X \longrightarrow Y$$
$$x \longmapsto f(x)$$

Example:

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2$$



## Advent of Mathematical Symbols

Little o:  $f(x) = o(g(x)) \quad (x \rightarrow a)$

means:  $\lim_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| = 0$

Example:  $8 \cdot x^2 \neq o(x^2) \quad (x \rightarrow \infty)$   
 $8 \cdot x^2 = o(x^3) \quad (x \rightarrow \infty)$



# Advent of Mathematical Symbols

Outer product  
(Kronecker product for vectors)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 \cdot w_1 & v_1 \cdot w_2 & v_1 \cdot w_3 \\ v_2 \cdot w_1 & v_2 \cdot w_2 & v_2 \cdot w_3 \end{pmatrix}$$

matrix entries:  $(v \otimes w)_{ij} = v_i w_j$



# Advent of Mathematical Symbols

Euler's phi function:  $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$   
 $\mathbb{N} = \{1, 2, 3, \dots\}$

Examples:  $\varphi(4) = 2$   $[1, \cancel{2}, 3, \cancel{4}]$

$\varphi(5) = 4$   $[1, 2, 3, 4, \cancel{5}]$

$\varphi(p) = p - 1$  for  $p$  prime

$\varphi(n) =$  count numbers  $a \in \mathbb{N}$  with

- (1)  $a \leq n$
- (2)  $\gcd(a, n) = 1$  (mutually prime)

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# The Bright Side of Mathematics



Day 20

## Advent of Mathematical Symbols

Laplace operator  
Laplacian

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \frac{\partial^2 f}{\partial x_3^2}(x)$$

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}$$



## Advent of Mathematical Symbols

Convolution  $(f * g)(x) := \int_{-\infty}^{\infty} f(\tau) \cdot g(x - \tau) d\tau$

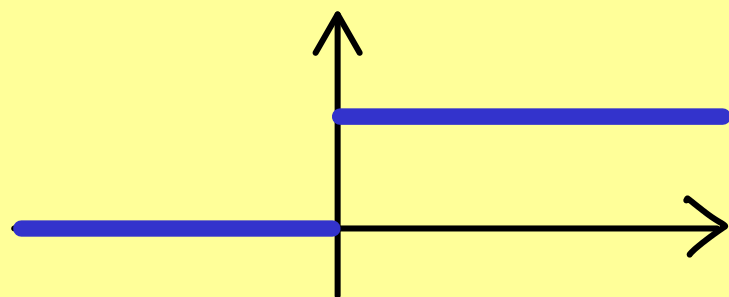
$f: \mathbb{R} \rightarrow \mathbb{R}$   
 $g: \mathbb{R} \rightarrow \mathbb{R}$  } new function:  $f * g: \mathbb{R} \rightarrow \mathbb{R}$



## Advent of Mathematical Symbols

Heaviside  
function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$H' = \delta$$



# Advent of Mathematical Symbols

Quaternions:

$$a, b, c, d \in \mathbb{R}$$

$$\mathbb{H} \supseteq \mathbb{C}$$

(William Rowan Hamilton)

↪ multiplication is not commutative

$$a + i \cdot b + j \cdot c + k \cdot d \quad , \quad i^2 = -1 \quad , \quad j^2 = -1 \quad , \quad k^2 = -1 \quad , \quad ijk = -1$$

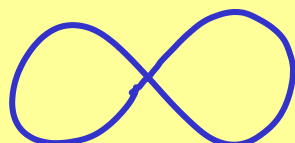
$$\Rightarrow ij = -ji$$



Day 24  
(Last)

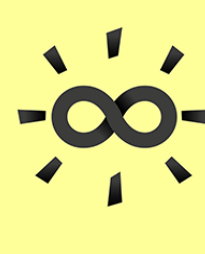
## Advent of Mathematical Symbols

Infinity:

For example:  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ In Measure Theory:  $[0, \infty]$ 

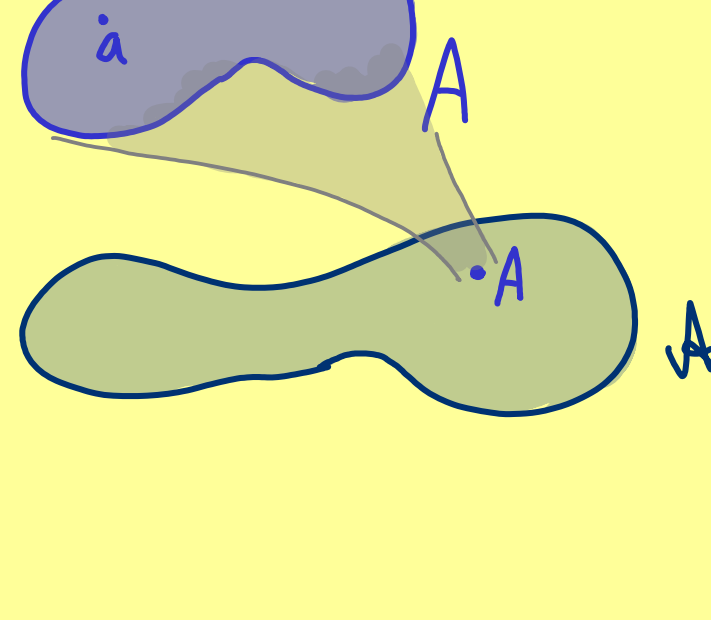
$$a + \infty = \infty + a = \infty \quad \text{for } a \in [0, \infty]$$

$$a \cdot \infty = \begin{cases} \infty & \text{for } a \in (0, \infty] \\ 0 & \text{for } a = 0 \end{cases}$$



# Advent of Mathematical Symbols

element of  $a \in A$   
 $A \in \mathcal{A}$



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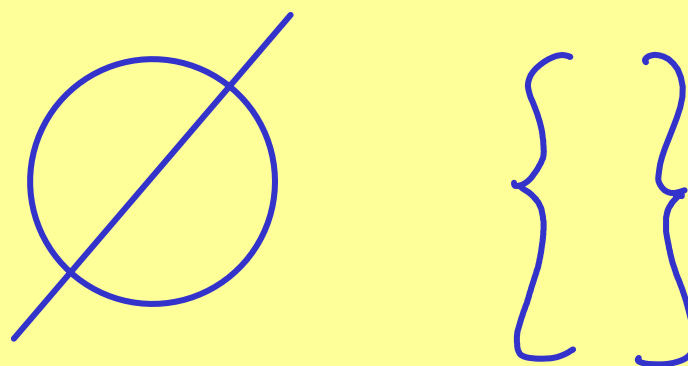
ON STEADY

# The Bright Side of Mathematics



Day 2  
(2022)

## Advent of Mathematical Symbols



For all  $x \in \emptyset$  holds:  $x$  even  $\Rightarrow$   $x$  odd

empty set  
=  
set with no elements  
 $x \in \emptyset$  false

BECOME A MEMBER

ON STEADY

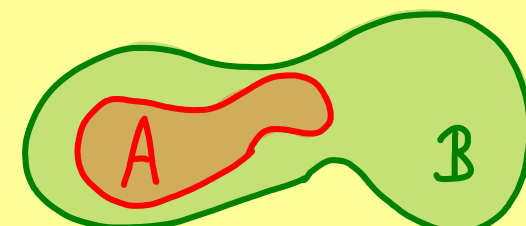
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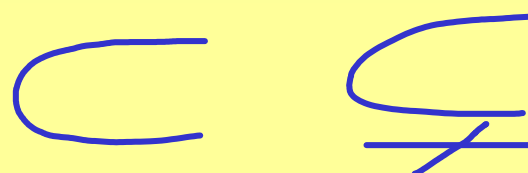
Day 3  
(2022)

## Advent of Mathematical Symbols

subset (equality included)



proper subset  
(equality excluded)



$$A \subseteq B, B \supseteq A$$

means:

for all  $x$ :  $x \in A \Rightarrow x \in B$

BECOME A MEMBER

ON STEADY

# The Bright Side of Mathematics



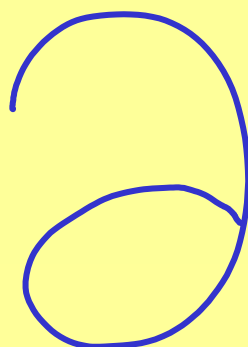
Day 4  
(2022)

## Advent of Mathematical Symbols

partial d

=

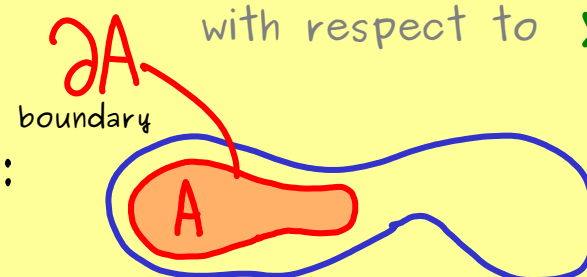
cursive d



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \frac{\partial f}{\partial x_1}$$

partial derivative of  $f$   
with respect to  $x_1$

topology:



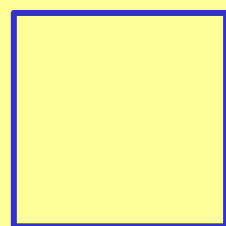


Day 5  
(2022)

## Advent of Mathematical Symbols

d'Alembert operator

- three dimensions in space
- one dimension in time



$$= \frac{\partial^2}{\partial t^2} -$$



$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\frac{1}{c^2}$$

Day 6  
(2022)

## Advent of Mathematical Symbols

inner product:

$$\langle \cdot, \cdot \rangle$$

map:  $V \times V \rightarrow \mathbb{C}$

vector space

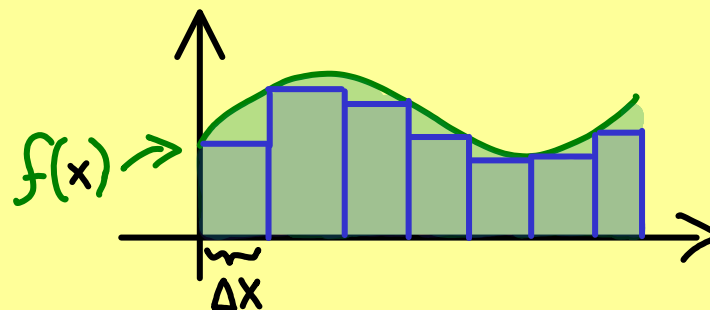
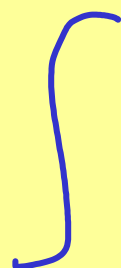
$$\langle \cdot | \cdot \rangle$$

in physics:  $\langle \psi | \tilde{\psi} \rangle$  bracket

bra:  $\langle \psi |$  ket:  $|\tilde{\psi}\rangle$

Day 7  
(2022)

## Advent of Mathematical Symbols

integral symbol:comes from sum

$$\sum f(x) \cdot \Delta x \xrightarrow{\text{limit}} \int f(x) dx$$



BECOME A MEMBER

ON STEADY

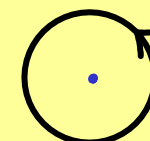
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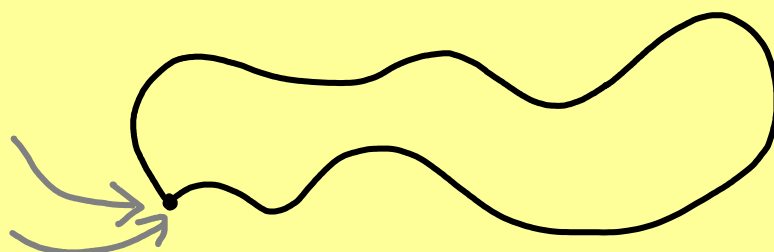
Day 8  
(2022)

## Advent of Mathematical Symbols

closed line integral



start  
||  
end



$$\oint \frac{1}{z} dz = 2\pi i$$

Day 9  
(2022)

## Advent of Mathematical Symbols

natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

or

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

together with addition +: monoid

→ associative:  $a + (b + c) = (a + b) + c$

→ neutral element:  $a + 0 = 0 + a = a$


 Day 10  
 (2022)

# Advent of Mathematical Symbols

integers

(whole numbers)

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

together with addition +:

group:

associative:  $a + (b + c) = (a + b) + c$

neutral element:  $a + 0 = 0 + a = a$

inverse elements:  $a + (-a) = (-a) + a = 0$