The Bright Side of Mathematics

The following pages cover the whole Advent of Mathematical Symbols course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Day 01

Advent of Mathematical Symbols

Kronecker delta: $$\delta_{ij} := \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}$$

Example: $\delta_{12} = 0$, $\delta_{55} = 1$, $\sum_{i,j=1}^{5} \delta_{ij} = 5$

Day 02

Advent of Mathematical Symbols

Levi-Civita symbol: $$\varepsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1, & (i,j,k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0, & \text{else} \end{cases}$$

Example: $$(a \times b)_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\, a_j\, b_k$$

Day 03

Advent of Mathematical Symbols

Nabla symbol: $$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

Example: $f(x_1,x_2) = x_1^3$, $\nabla f(x_1,x_2) = \begin{pmatrix} 3x_1^2 \\ 0 \end{pmatrix}$

Advent of Mathematical Symbols

Factorial: $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

Example: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $1! = 1$

Recursive definition: $0! := 1$, $n! := n \cdot (n-1)!$ $(n \in \mathbb{N})$

Advent of Mathematical Symbols

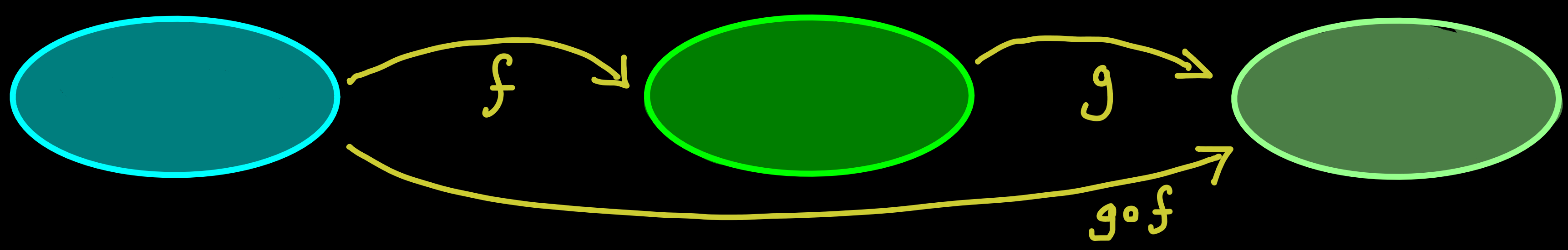
Gamma function: $\Gamma(z) := \int_0^\infty x^{z-1} \cdot e^{-x} \, dx \quad , \; \mathrm{Re}(z) > 0$

Property: $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, $\Gamma(z+1) = z \cdot \Gamma(z)$

Day 06

Advent of Mathematical Symbols

Composition: $$(g \circ f)(x) := g(f(x))$$



Day 07

Advent of Mathematical Symbols

sum:

$$\sum_{k=1}^{n} a_k := a_1 + a_2 + \cdots + a_n$$

recursive definition: $$\sum_{k=1}^{0} a_k := 0 \quad , \quad \sum_{k=1}^{n} a_k := \left(\sum_{k=1}^{n-1} a_k \right) + a_n$$

Advent of Mathematical Symbols

product: $$\prod_{k=1}^{n} a_k := a_1 \cdot a_2 \cdot \; \cdots \; \cdot a_n$$

recursive definition: $$\prod_{k=1}^{0} a_k := 1 \quad , \qquad \prod_{k=1}^{n} a_k := \left(\prod_{k=1}^{n-1} a_k \right) \cdot a_n$$

Day 09

Advent of Mathematical Symbols

restriction: $f|_A : A \longrightarrow Y$

For $f : X \to Y$ and $A \subseteq X$: $f|_A(x) \overset{\text{for all } x \in A}{=} f(x)$

Day 10

Advent of Mathematical Symbols

Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We have: $\sigma_k^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_j \sigma_k - \sigma_k \sigma_j = 2i\, \varepsilon_{jk\ell} \sigma_\ell$

Advent of Mathematical Symbols

set brackets: $\{ f(x) \mid x \in A \}$

Example: $\{ 2 \cdot x + 1 \mid x \in \{0, 1, 2, 3\} \} = \{1, 3, 5, 7\}$

Day 12

Advent of Mathematical Symbols

Big O: $f(x) = \mathcal{O}(g(x)) \quad (x \to a)$

means: $|f(x)| \leq M \cdot |g(x)|$

$\left(\limsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty \right)$

Example: $x^2 + x + 2 = \mathcal{O}(x^2) \ (x \to \infty)$

$x^2 + x + 2 = \mathcal{O}(x^3) \ (x \to \infty)$

Day 13

Advent of Mathematical Symbols

Binomial coefficient: $\binom{n}{k} = \dfrac{n\cdot(n-1)\cdots(n-k+1)}{k!} = \dfrac{n!}{k!(n-k)!}$

1 2 3 4 5 6 7

$n = 7$

Take: $k = 3$

2 3 6

$$\frac{n\cdot(n-1)\cdot(n-2)}{3\cdot 2\cdot 1}$$

or 5 6 7 ...

Advent of Mathematical Symbols

Modulo: $x \bmod n := r \in [0, n)$

with $x = n \cdot q + r$ ← integer

Examples:

$5 \bmod 3 = 2$

$6 \bmod 3 = 0$

$7.1 \bmod 3 = 1.1$

$9.7 \bmod 2.1 = 1.3$

$9.7 \xrightarrow{-2.1} 7.6 \xrightarrow{-2.1} 5.5 \xrightarrow{-2.1} 3.4 \xrightarrow{-2.1} 1.3$

Advent of Mathematical Symbols

Beta function: $$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$x, y \in \mathbb{C}$,
$\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$

$$B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$$

Advent of Mathematical Symbols

$$\begin{aligned} f: X &\longrightarrow Y \\ x &\longmapsto f(x) \end{aligned}$$

Example:

$$\begin{aligned} f: \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

Day 17

Advent of Mathematical Symbols

Little o: $f(x) = o(g(x)) \quad (x \to a)$

means: $$\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0$$

Example: $8 \cdot x^2 \neq o(x^2) \quad (x \to \infty)$

$8 \cdot x^2 = o(x^3) \quad (x \to \infty)$

Advent of Mathematical Symbols

Outer product

(Kronecker product for vectors)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 \cdot w_1 & v_1 \cdot w_2 & v_1 \cdot w_3 \\ v_2 \cdot w_1 & v_2 \cdot w_2 & v_2 \cdot w_3 \end{pmatrix}$$

matrix entries: $(v \otimes w)_{ij} = v_i w_j$

Advent of Mathematical Symbols

Euler's phi function: $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

Examples: $\varphi(4) = 2$ $[1, \cancel{2}, 3, \cancel{4}]$

$\varphi(5) = 4$ $[1, 2, 3, 4, \cancel{5}]$

$\varphi(p) = p - 1$ for p prime

$\varphi(n) =$ count numbers $a \in \mathbb{N}$ with

(1) $a \leq n$

(2) $\gcd(a, n) = 1$ (mutually prime)

Day 20

Advent of Mathematical Symbols

Laplace operator

Laplacian

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \frac{\partial^2 f}{\partial x_3^2}(x)$$

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

Advent of Mathematical Symbols

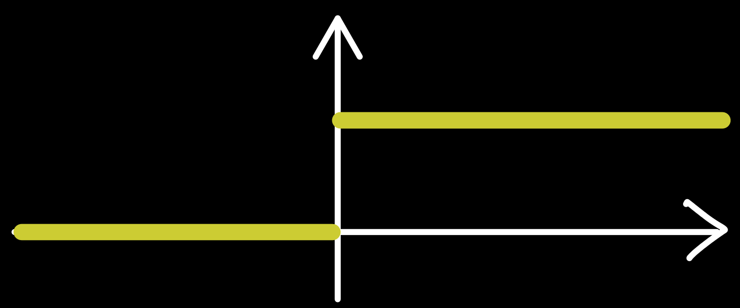
Convolution $(f * g)(x) := \int_{-\infty}^{\infty} f(\tau) \cdot g(x-\tau)\, d\tau$

$$\left.\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ g &: \mathbb{R} \to \mathbb{R} \end{aligned}\right\} \text{new function: } f * g : \mathbb{R} \to \mathbb{R}$$

Advent of Mathematical Symbols

Heaviside function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$H' = \delta$$

Day 23

Advent of Mathematical Symbols

Quaternions: $\mathbb{H} \supseteq \mathbb{C}$ (William Rowan Hamilton)

$a, b, c, d \in \mathbb{R}$

multiplication is not commutative

$$a + i \cdot b + j \cdot c + k \cdot d \quad , \quad i^2 = -1 \; , \; j^2 = -1 \; , \; k^2 = -1 \quad , \quad ijk = -1$$

$$\Longrightarrow \; i \cdot j = -j \cdot i$$

Day 24
(Last)

Advent of Mathematical Symbols

Infinity: ∞

For example: $\lim_{n \to \infty} \frac{1}{n} = 0$

In Measure Theory: $[0, \infty]$

$a + \infty = \infty + a = \infty$ for $a \in [0, \infty]$

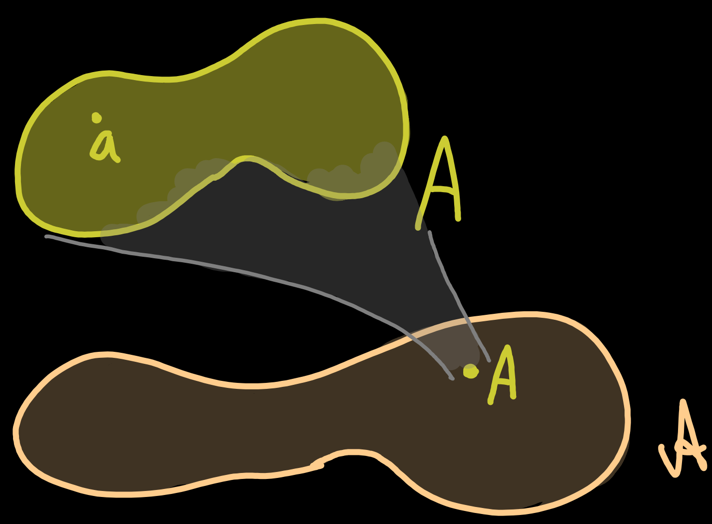
$$a \cdot \infty = \begin{cases} \infty & \text{for } a \in (0, \infty] \\ 0 & \text{for } a = 0 \end{cases}$$

Day 1
(2022)

Advent of Mathematical Symbols

element of $a \in A$

$A \in \mathcal{A}$



Advent of Mathematical Symbols

Day 2
(2022)

$$\emptyset \qquad \{\ \}$$

empty set
=
set with no elements

For all $x \in \emptyset$ holds: x even $\Rightarrow$ x odd

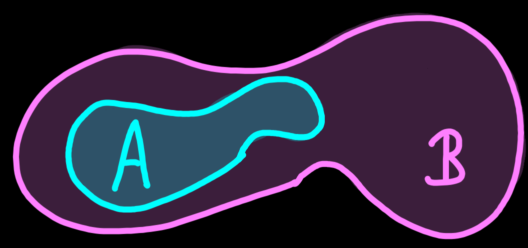
$x \in \emptyset$ false

Advent of Mathematical Symbols

Day 3 (2022)

subset (equality included) $\subseteq$ $\subset$

proper subset (equality excluded) $\subset$ $\subsetneq$



$A \subseteq B$, $B \supseteq A$

means:

for all x : $x \in A \Rightarrow x \in B$

Advent of Mathematical Symbols

Day 4 (2022)

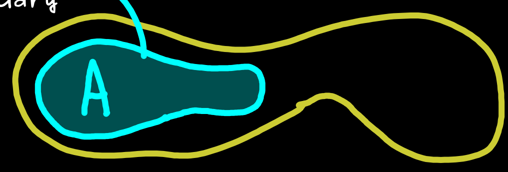
partial d = cursive d

$$\partial$$

$f: \mathbb{R}^2 \to \mathbb{R}$, $\dfrac{\partial f}{\partial x_1}$

partial derivative of f with respect to x_1

topology: ∂A boundary



Day 5 (2022)

Advent of Mathematical Symbols

d'Alembert operator

- three dimensions in space
- one dimension in time

$$\Box = \underbrace{\frac{1}{c^2}}\frac{\partial^2}{\partial t^2} - \underbrace{\Delta}_{\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}}$$

Day 6 (2022)

Advent of Mathematical Symbols

inner product: $\langle \cdot , \cdot \rangle$ $\qquad$ $\langle \cdot | \cdot \rangle$

map: $V \times V \longrightarrow \mathbb{C}$

vector space

in physics: $\langle \psi | \tilde{\psi} \rangle$ bracket

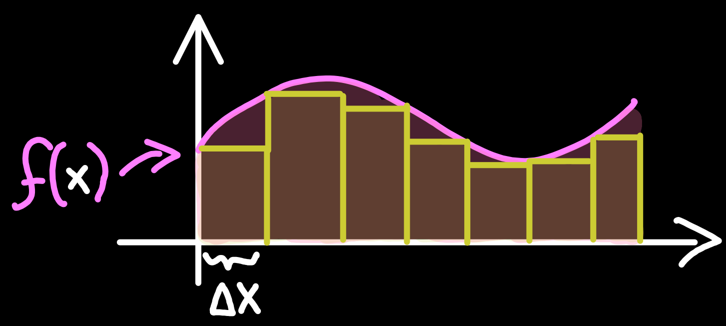
bra: $\langle \psi |$ $\qquad$ ket: $| \tilde{\psi} \rangle$

Day 7
(2022)

Advent of Mathematical Symbols

integral symbol: $$\int$$

comes from sum



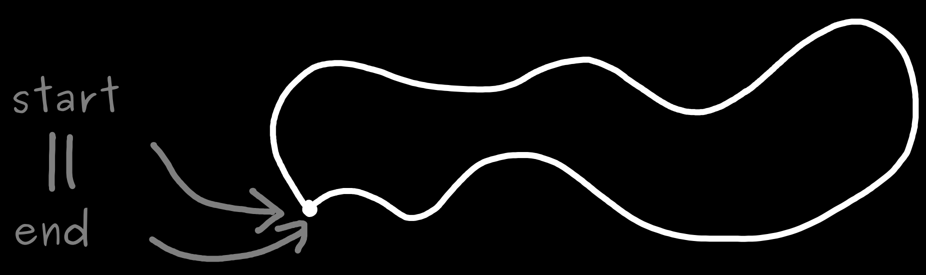
$$\sum f(x) \cdot \Delta x \xrightarrow{\text{limit}} \int f(x)\, dx$$

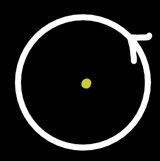
Day 8
(2022)

Advent of Mathematical Symbols

closed line integral

$$\oint$$





$$\oint \frac{1}{z}\,dz = 2\pi i$$

Day 9 (2022)

Advent of Mathematical Symbols

natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ or

$= \{0, 1, 2, 3, 4, \ldots\}$

together with addition $+$: monoid

associative: $a + (b + c) = (a + b) + c$

neutral element: $a + 0 = 0 + a = a$

Day 10
(2022)

Advent of Mathematical Symbols

integers

(whole numbers)

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

together with addition $+$: **group:**

associative: $a + (b + c) = (a + b) + c$

neutral element: $a + 0 = 0 + a = a$

inverse elements: $a + (-a) = (-a) + a = 0$