

The Bright Side of Mathematics

The following pages cover the whole Advent of Mathematical Symbols course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: <https://tbsom.de/support>

Have fun learning mathematics!

Day 01

Advent of Mathematical Symbols

Kronecker delta: $\delta_{ij} := \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}$

Example: $\delta_{12} = 0$, $\delta_{55} = 1$, $\sum_{i,j=1}^5 \delta_{ij} = 5$

Day 02

Advent of Mathematical Symbols

Levi-Civita symbol: $\epsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1, & (i,j,k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0, & \text{else} \end{cases}$

Example: $(a \times b)_i = \sum_{j,k=1}^3 \epsilon_{ijk} a_j b_k$

Day 03

Advent of Mathematical Symbols

Nabla symbol: $\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$ or $\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$

Example: $f(x_1, x_2) = x_1^3$, $\nabla f(x_1, x_2) = \begin{pmatrix} 3x_1^2 \\ 0 \end{pmatrix}$

Day 04

Advent of Mathematical Symbols

Factorial: $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

Example: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $1! = 1$

Recursive definition: $0! := 1$, $n! := n \cdot (n-1)! \quad (n \in \mathbb{N})$

Day 05

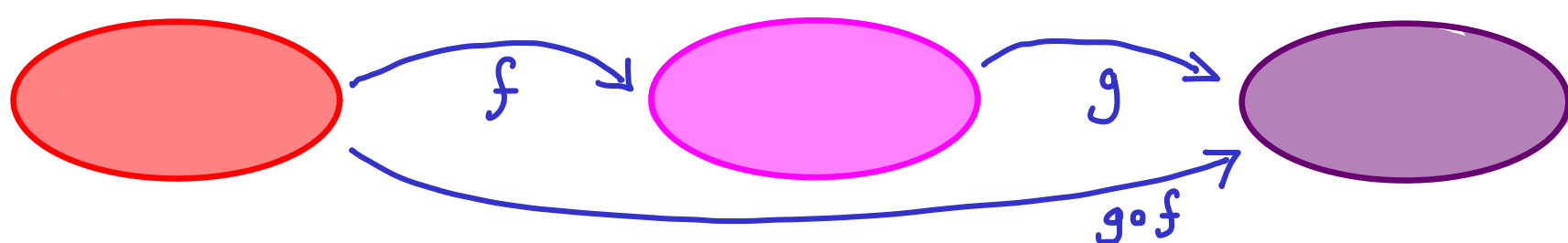
Advent of Mathematical Symbols

Gamma function: $\Gamma(z) := \int_0^{\infty} x^{z-1} \cdot e^{-x} dx$, $\operatorname{Re}(z) > 0$

Property: $\Gamma(n) = (n-1)!$, $\Gamma(z+1) = z \cdot \Gamma(z)$
for $n \in \mathbb{N}$

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Composition: $(g \circ f)(x) := g(f(x))$



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sum:

$$\sum_{k=1}^n a_k := a_1 + a_2 + \dots + a_n$$

recursive definition:

$$\sum_{k=1}^0 a_k := 0 \quad , \quad \sum_{k=1}^n a_k := \left(\sum_{k=1}^{n-1} a_k \right) + a_n$$

Advent of Mathematical Symbols

product: $\prod_{k=1}^n a_k := a_1 \cdot a_2 \cdot \dots \cdot a_n$

recursive definition: $\prod_{k=1}^0 a_k := 1$, $\prod_{k=1}^n a_k := \left(\prod_{k=1}^{n-1} a_k \right) \cdot a_n$

Advent of Mathematical Symbols

restriction: $f|_A : A \rightarrow Y$

For $f: X \rightarrow Y$ and $A \subseteq X$: $f|_A(x) = f(x)$ for all $x \in A$

Advent of Mathematical Symbols

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have:

$$\sigma_k^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_j \sigma_k - \sigma_k \sigma_j = 2i \varepsilon_{jkl} \sigma_l$$

Advent of Mathematical Symbols

set brackets: $\{ f(x) \mid x \in A \}$

Example: $\{ 2 \cdot x + 1 \mid x \in \{0, 1, 2, 3\} \} = \{1, 3, 5, 7\}$

Advent of Mathematical Symbols

$$\text{Big } O: f(x) = \mathcal{O}(g(x)) \quad (x \rightarrow a)$$

means: $|f(x)| \leq M \cdot |g(x)|$

$$\left(\limsup_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| < \infty \right)$$

Example: $x^2 + x + 2 = \mathcal{O}(x^2) \quad (x \rightarrow \infty)$

$$x^2 + x + 2 = \mathcal{O}(x^3) \quad (x \rightarrow \infty)$$

Advent of Mathematical Symbols

Binomial coefficient: $\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$

Take: $k=3$ $\textcircled{1} \textcircled{2} \textcircled{3} \textcircled{4} \textcircled{5} \textcircled{6} \textcircled{7}$

or $\textcircled{5} \textcircled{6} \textcircled{7} \dots$

$n=7$

$\frac{n \cdot (n-1) \cdot (n-2)}{3 \cdot 2 \cdot 1}$

Advent of Mathematical Symbols

Modulo: $x \bmod n := r \in [0, n)$

with $x = n \cdot q + r$
↖ integer

Examples: $5 \bmod 3 = 2$

$6 \bmod 3 = 0$

$7.1 \bmod 3 = 1.1$

$9.7 \bmod 2.1 = 1.3$

$\begin{array}{r}
 \downarrow -2.1 \\
 7.6 \\
 \downarrow -2.1 \\
 5.5 \rightarrow 3.4 \rightarrow 1.3
 \end{array}$

Advent of Mathematical Symbols

Beta function: $B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt$

$$B(x, y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$$

$$x, y \in \mathbb{C},$$

$$\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$$

Advent of Mathematical Symbols

$$f: X \longrightarrow Y$$
$$x \longmapsto f(x)$$

Example: $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto x^2$$

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Little o: $f(x) = o(g(x)) \quad (x \rightarrow a)$

means: $\lim_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| = 0$

Example: $8 \cdot x^2 \neq o(x^2) \quad (x \rightarrow \infty)$
 $8 \cdot x^2 = o(x^3) \quad (x \rightarrow \infty)$

Advent of Mathematical Symbols

Outer product

(Kronecker product for vectors)

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \otimes \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} V_1 \cdot W_1 & V_1 \cdot W_2 & V_1 \cdot W_3 \\ V_2 \cdot W_1 & V_2 \cdot W_2 & V_2 \cdot W_3 \end{pmatrix}$$

$$\text{matrix entries: } (V \otimes W)_{ij} = V_i W_j$$

Advent of Mathematical Symbols

Euler's phi function: $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$
 $\mathbb{N} = \{1, 2, 3, \dots\}$

Examples: $\varphi(4) = 2$ $[1, 2, 3, 4]$

$\varphi(5) = 4$ $[1, 2, 3, 4, 5]$

$\varphi(p) = p - 1$ for p prime

$\varphi(n) =$ count numbers $a \in \mathbb{N}$ with

(1) $a \leq n$

(2) $\gcd(a, n) = 1$ (mutually prime)

Day 20

Advent of Mathematical Symbols

Laplace operator
Laplacian

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \frac{\partial^2 f}{\partial x_3^2}(x)$$

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

Advent of Mathematical Symbols

Convolution $(f * g)(x) := \int_{-\infty}^{\infty} f(\tau) \cdot g(x - \tau) d\tau$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

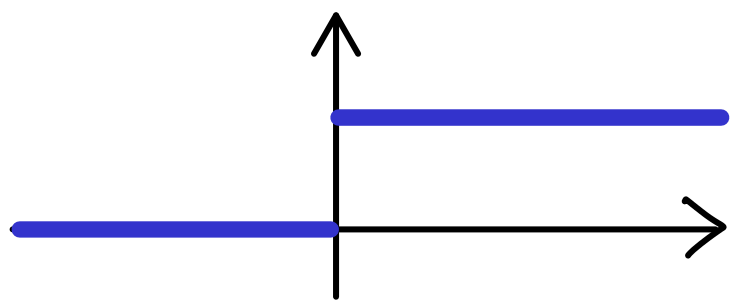
$$g: \mathbb{R} \rightarrow \mathbb{R}$$

} new function: $f * g: \mathbb{R} \rightarrow \mathbb{R}$

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Heaviside
function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$H' = \delta$$

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Quaternions: $\mathbb{H} \supseteq \mathbb{C}$ (William Rowan Hamilton)

$$a, b, c, d \in \mathbb{R}$$

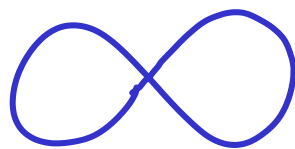
↶ multiplication is not commutative

$$a + i \cdot b + j \cdot c + k \cdot d, \quad i^2 = -1, \quad j^2 = -1, \quad k^2 = -1, \quad ijk = -1$$

$$\Rightarrow ij = -j \cdot i$$

Advent of Mathematical Symbols

Infinity:



For example: $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

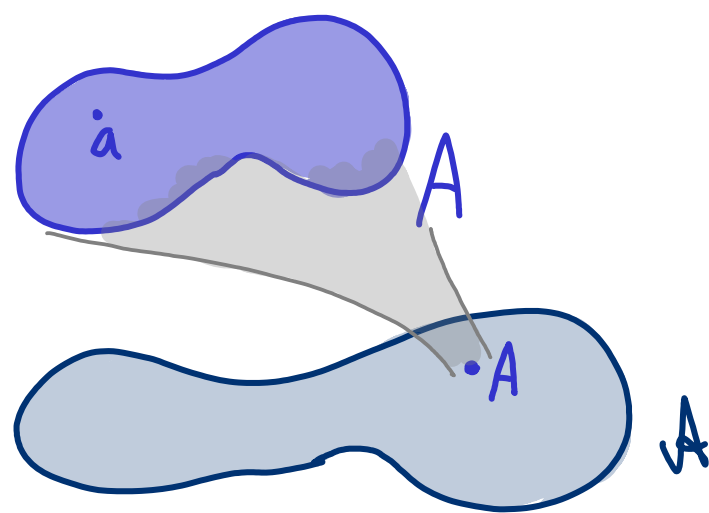
In Measure Theory: $[0, \infty]$

$$a + \infty = \infty + a = \infty \quad \text{for } a \in [0, \infty]$$

$$a \cdot \infty = \begin{cases} \infty & \text{for } a \in (0, \infty] \\ 0 & \text{for } a = 0 \end{cases}$$

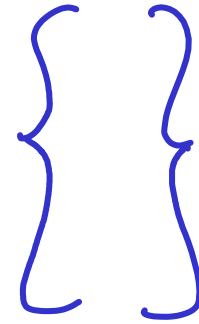
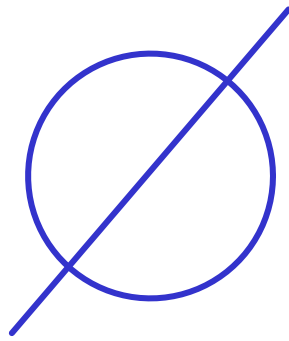
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element of $a \in A$
 $A \in \mathcal{A}$



Day 2
(2022)

Advent of Mathematical Symbols



empty set

=

set with no elements

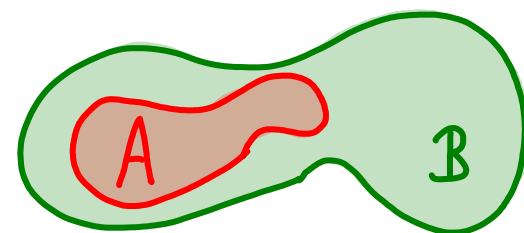
For all $x \in \emptyset$ holds: x even \Rightarrow x odd

$x \in \emptyset$ false

Advent of Mathematical Symbols

Day 3
(2022)

subset (equality included)



proper subset
(equality excluded)



$$A \subseteq B, B \supseteq A$$

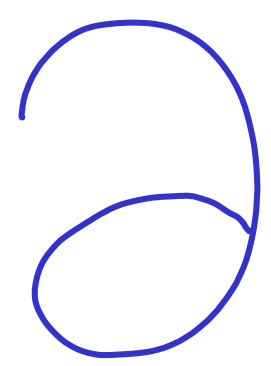
means:

$$\text{for all } x: x \in A \Rightarrow x \in B$$

Day 4
(2022)

Advent of Mathematical Symbols

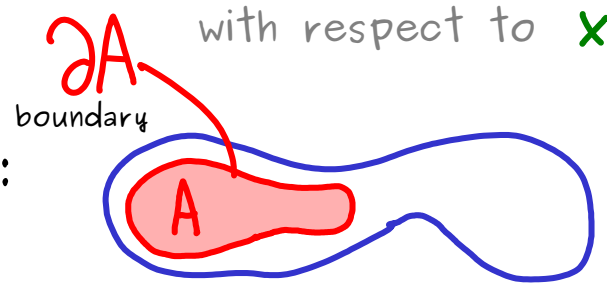
partial d
=
cursive d



$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \frac{\partial f}{\partial x_1}$$

partial derivative of f
with respect to x_1

topology:

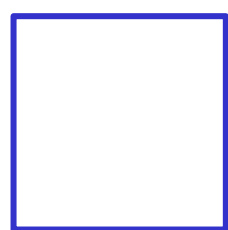


Day 5
(2022)

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d'Alembert operator

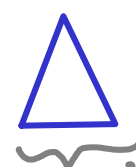
- three dimensions in space
- one dimension in time



$$= \frac{\partial^2}{\partial t^2}$$

$$\frac{1}{c^2}$$

-



$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

Day 6
(2022)

Advent of Mathematical Symbols

inner product:

$$\langle \cdot, \cdot \rangle$$

$$\langle \cdot | \cdot \rangle$$

map: $V \times V \rightarrow \mathbb{C}$

vector space

in physics: $\langle \psi | \tilde{\psi} \rangle$ bracket

bra:

$$\langle \psi |$$

ket:

$$| \tilde{\psi} \rangle$$

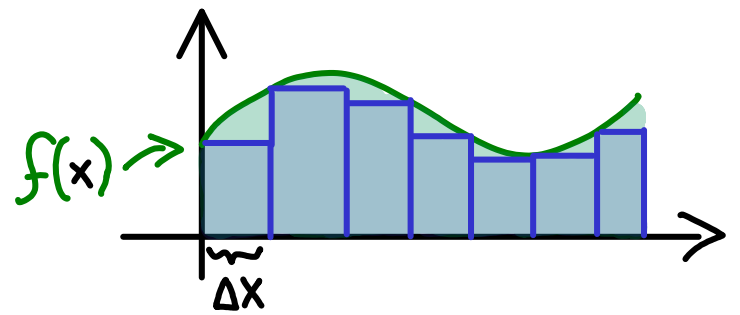
Day 7
(2022)

Advent of Mathematical Symbols

integral symbol:

comes from sum

\int

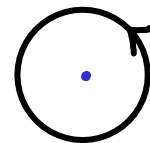


$$\sum f(x) \cdot \Delta x \xrightarrow{\text{limit}} \int f(x) dx$$

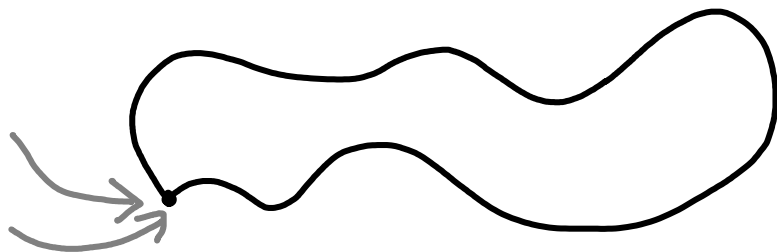
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Day 8
(2022)

closed line integral



start
||
end



$$\oint \frac{1}{z} dz = 2\pi i$$

Day 9
(2022)

Advent of Mathematical Symbols

natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
 $= \{0, 1, 2, 3, 4, \dots\}$ or

together with addition +: monoid \rightarrow associative: $a + (b + c) = (a + b) + c$
 \rightarrow neutral element: $a + 0 = 0 + a = a$

Advent of Mathematical Symbols

integers

(whole numbers)

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$



associative: $a + (b + c) = (a + b) + c$

neutral element: $a + 0 = 0 + a = a$

inverse elements: $a + (-a) = (-a) + a = 0$

together with addition +: group: