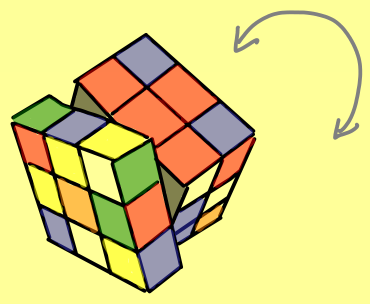
Algebra – Part 4

$(S, \circ)$ semigroup $\rightsquigarrow$ + neutral element, inverses $\rightsquigarrow$ group



Definition: A pair $(G, \circ)$ is called a group if:

(a) $(G, \circ)$ semigroup.

(b) There is a left identity $e \in G$.

(c) Each $a \in G$ is left invertible, i.e. there exists $b \in G$ with $b \circ a = e$.

This implies: A set G together with a binary operation $\circ$ is a group if:

(G1) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$ (associative)

(G2) There is a unique identity $e \in G$: $e \circ a = a = a \circ e$ for all $a \in G$

(G3) Each $a \in G$ is invertible: $\exists\, b \in G$: $b \circ a = e = a \circ b$

$a^{-1} := b$ (common notation)

Proof: (a) $\Rightarrow$ (G1) ✓

Let $a \in G$.

(b) There is a left identity $e \in G$.

(c) Each $a \in G$ is left invertible, i.e. there exists $b \in G$ with $b \circ a = e$. (*)

Choose $b \in G$ with $ba = e$. Then $ab \overset{(b)}{=} a(eb) \overset{(*)}{=} a(ba)b = (ab)(ab)$. (**)

Choose $c \in G$ with $c(ab) = e$ (by (c))

$\Rightarrow ab \overset{(b)}{=} e(ab) = c(ab)(ab) \overset{(**)}{=} c(ab) = e \Rightarrow$ (G3) ✓

$\Rightarrow ae \overset{(*)}{=} a(ba) = (ab)a = ea = a \Rightarrow$ (G2) ✓