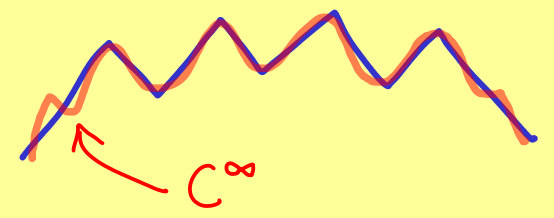


# The Bright Side of Mathematics



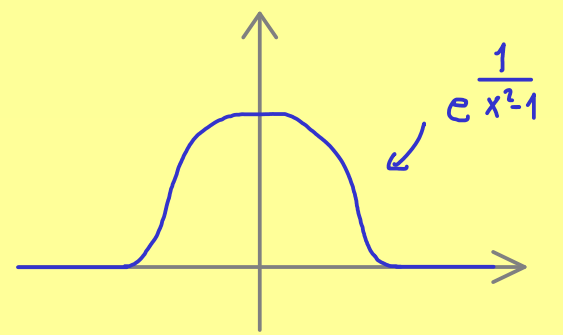
## Approximation Theorem



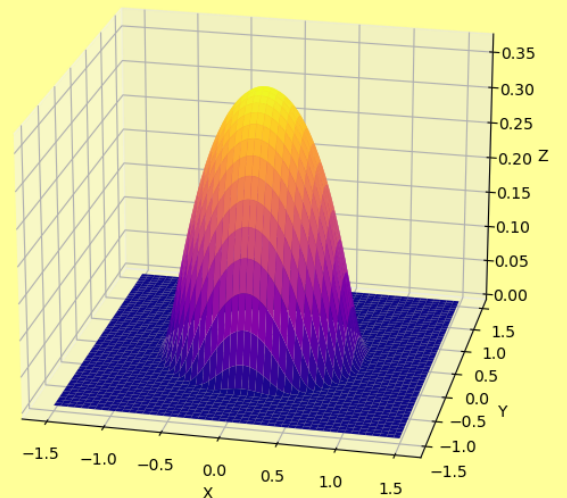
Standard mollifier:  $C^\infty$ -function with compact support:

$$\eta: \mathbb{R}^n \rightarrow \mathbb{R}$$

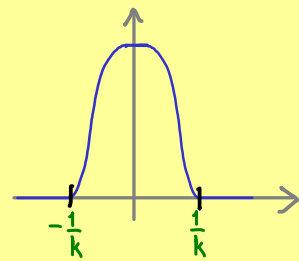
$$\eta(x) := \begin{cases} C \cdot \exp\left(\frac{1}{\|x\|^2-1}\right) & , \|x\| < 1 \\ 0 & , \|x\| \geq 1 \end{cases}$$



where  $C > 0$  such that  $\int_{\mathbb{R}^n} \eta(x) d^n x = 1$ .



Define a Dirac sequence:  $\delta_k(x) := k^n \cdot \eta(k \cdot x)$



with three properties:

(1)  $\delta_k(x) \geq 0$  for all  $k \in \mathbb{N}$ , for all  $x \in \mathbb{R}^n$

(2)  $\int_{\mathbb{R}^n} \delta_k(x) d^n x = 1$  for all  $k \in \mathbb{N}$

(3) For all  $\epsilon > 0$ ,  $\mathcal{B}_\epsilon(0) := \{y \in \mathbb{R}^n \mid \|y\| < \epsilon\}$  satisfies:

$$\int_{\mathbb{R}^n \setminus \mathcal{B}_\epsilon(0)} \delta_k(x) d^n x \xrightarrow{k \rightarrow \infty} 0$$

Approximation Theorem: Let  $f: \mathbb{R}^n \rightarrow \mathbb{C}$  be continuous.

Then for each compact set  $A \subseteq \mathbb{R}^n$ :  $\|f - \underbrace{\delta_k * f}_{C^\infty}\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$   
 $(\delta_k(x) := k^n \cdot \eta(k \cdot x))$   $\underbrace{\qquad\qquad\qquad}_{C^\infty} = \sup_{x \in A} |\cdot|$

Proof: Let  $\epsilon > 0$  and  $A \subseteq \mathbb{R}^n$  be compact.

So we find  $\delta > 0$  such that for all  $x \in A$  and  $y \in \mathcal{B}_\delta(0)$ :  $|f(x-y) - f(x)| < \epsilon$

$f|_A$  uniformly continuous  
 $\forall \epsilon > 0 \exists \delta > 0 \forall x, \tilde{x} \in A:$   
 $\|\tilde{x} - x\| < \delta \Rightarrow |f(\tilde{x}) - f(x)| < \epsilon$

Then for any  $x \in A$ :

$$\begin{aligned} |f(x) - (\delta_k * f)(x)| &= \left| f(x) - \int_{\mathbb{R}^n} \delta_k(y) f(x-y) d^n y \right| \\ &= \left| f(x) \underbrace{\int_{\mathbb{R}^n} \delta_k(y) d^n y}_{=1} - \int_{\mathbb{R}^n} \delta_k(y) f(x-y) d^n y \right| \\ &= \left| \int_{\mathbb{R}^n} \delta_k(y) (f(x) - f(x-y)) d^n y \right| \stackrel{(1)}{\leq} \int_{\mathbb{R}^n} \delta_k(y) |f(x-y) - f(x)| d^n y \\ &= \int_{\mathcal{B}_{\frac{\epsilon}{2}}(0)} \delta_k(y) \underbrace{|f(x-y) - f(x)|}_{< \epsilon} d^n y + \int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\epsilon}{2}}(0)} \delta_k(y) |f(x-y) - f(x)| d^n y \\ &\leq \epsilon + \int_{\mathbb{R}^n \setminus \mathcal{B}_{\frac{\epsilon}{2}}(0)} \delta_k(y) (|f(x-y)| + |f(x)|) d^n y \\ &\qquad\qquad\qquad = 0 \text{ for all } k \geq K \end{aligned}$$

$$\Rightarrow \sup_{x \in A} |f(x) - (\delta_k * f)(x)| \leq \epsilon \Rightarrow \|f - \delta_k * f\|_{\infty, A} \xrightarrow{k \rightarrow \infty} 0$$

□