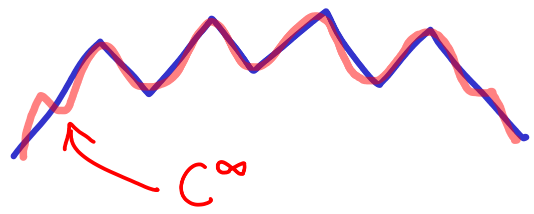
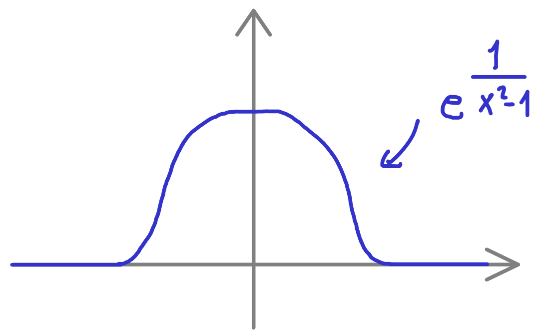
Approximation Theorem

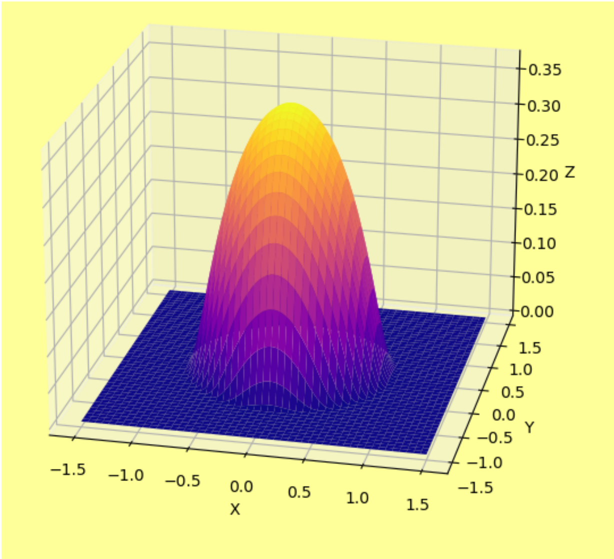


Standard mollifier: C^∞-function with compact support:



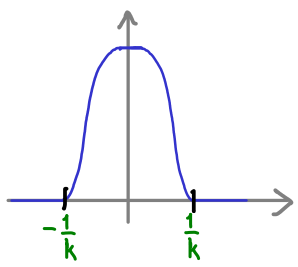
$$\eta : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\eta(x) := \begin{cases} c \cdot \exp\left(\frac{1}{\|x\|^2-1}\right) & , \ \|x\| < 1 \\ 0 & , \ \|x\| \geq 1 \end{cases}$$



where $c > 0$ such that $\int_{\mathbb{R}^n} \eta(x)\, d^n x = 1$.

Define a Dirac sequence: $\delta_k(x) := k^n \cdot \eta(k \cdot x)$



with three properties:

(1) $\delta_k(x) \geq 0$ for all $k \in \mathbb{N}$, for all $x \in \mathbb{R}^n$

(2) $\int_{\mathbb{R}^n} \delta_k(x)\, d^n x = 1$ for all $k \in \mathbb{N}$

(3) For all $\varepsilon > 0$, $B_\varepsilon(0) := \{ y \in \mathbb{R}^n \mid \|y\| < \varepsilon \}$ satisfies:

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \delta_k(x)\, d^n x \xrightarrow{k \to \infty} 0$$

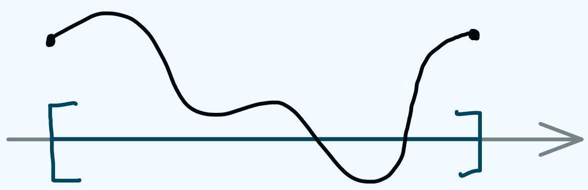
Approximation Theorem: Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be continuous.

Then for each compact set $A \subseteq \mathbb{R}^n$: $\| f - \underbrace{\delta_k * f}_{C^\infty} \|_{\infty, A} \xrightarrow{k \to \infty} 0$, where $\|\cdot\|_{\infty,A} = \sup_{x \in A} |\cdot|$

$$\left(\delta_k(x) := k^n \cdot \eta(k \cdot x) \right)$$

Proof: Let $\varepsilon > 0$ and $A \subseteq \mathbb{R}^n$ be compact.

So we find $\delta > 0$ such that for all $x \in A$ and $y \in B_\delta(0)$: $|f(x-y) - f(x)| < \varepsilon$



$f|_A$ uniformly continuous

$\forall \varepsilon > 0 \;\exists \delta > 0 \;\forall x, \tilde{x} \in A: \; \|\tilde{x} - x\| < \delta \Rightarrow |f(\tilde{x}) - f(x)| < \varepsilon$

Then for any $x \in A$:

$$|f(x) - (\delta_k * f)(x)| = \left| f(x) - \int_{\mathbb{R}^n} \delta_k(y) f(x-y)\, d^n y \right|$$

$$= \left| f(x) \underbrace{\int_{\mathbb{R}^n} \delta_k(y)\, d^n y}_{\overset{(2)}{=} 1} - \int_{\mathbb{R}^n} \delta_k(y) f(x-y)\, d^n y \right|$$

$$= \left| \int_{\mathbb{R}^n} \delta_k(y) \big(f(x) - f(x-y) \big)\, d^n y \right| \leq \int_{\mathbb{R}^n} \overbrace{\delta_k(y)}^{(1)} |f(x-y) - f(x)|\, d^n y$$

$$= \int_{B_{\frac{\delta}{2}}(0)} \delta_k(y) \underbrace{|f(x-y) - f(x)|}_{< \varepsilon}\, d^n y + \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}(0)} \delta_k(y) |f(x-y) - f(x)|\, d^n y$$

$$\leq \varepsilon + \underbrace{\int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}(0)} \delta_k(y) \big(|f(x-y)| + |f(x)| \big)\, d^n y}_{= 0 \text{ for all } k \geq K}$$

$$\Longrightarrow \sup_{x \in A} |f(x) - (\delta_k * f)(x)| \leq \varepsilon \Longrightarrow \| f - \delta_k * f \|_{\infty, A} \xrightarrow{k \to \infty} 0$$

$\square$