

Baire Category Theorem

(X, d) complete metric space

A countable intersection of dense open sets is dense!

$\hookrightarrow Q_j \subseteq X$ dense ($\overline{Q_j} = X$) + open
 $\implies \bigcap_{j=1}^{\infty} Q_j$ is also dense

Notions: Let (X, \mathcal{J}) be a topological space.

(1) $M \subseteq X$ nowhere dense (in X) if $(\overline{M})^\circ = \emptyset$

(2) $M \subseteq X$ meagre (in X)
 (set of first category (in X)) $\iff M = \bigcup_{j=1}^{\infty} S_j$ \leftarrow nowhere dense

(3) $M \subseteq X$ nonmeagre (in X)
 (set of second category (in X)) $\iff M$ is not meagre

Baire category theorem: (X, d) complete metric space

\implies each open set ($\neq \emptyset$) is of second category (in X)

special version: (X, d) complete metric space $\implies X$ is of second category (in X)

Application: $C([0,1]) = \underbrace{\bigcup_{j=1}^{\infty} A_j}_{\cong \{ \text{functions that are differentiable at least at one point in } [0,1] \}} \cup \underbrace{B}_{\text{nowhere dense}} \subseteq \{ \text{functions that are nowhere differentiable} \}$

Baire
 $\implies B \neq \emptyset$ nonmeagre, B is dense in $C([0,1])$