



# Linear Algebra

## Eigenvalues and eigenvectors

Exercise 1. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 5 & 6 \\ 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 6 \\ 0 & 5 \end{bmatrix}$$



$$Av = \lambda v \quad v \neq 0$$
$$(A - I\lambda)v = \vec{0} \quad \det(A - I\lambda) = 0$$

$$\begin{vmatrix} 5-\lambda & 6 \\ 0 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 = 0 \quad \lambda = 5$$

$$\begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} 6x_2 = 0 \\ x_2 = 0 \end{matrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t, \quad t \in \mathbb{R} \setminus \{0\}$$

$$B = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = + (1-\lambda) [(-2-\lambda)(4-\lambda) + 8]$$

$$\lambda_1 = 0$$

$$\begin{pmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= (1-\lambda) [-8 + 2\lambda - 4\lambda + \lambda^2 + 8] = (1-\lambda)(-2\lambda + \lambda^2)$$

$$= \lambda(1-\lambda)(\lambda-2) = 0$$

$$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$$

$$\begin{pmatrix} 0 & 0 & -4 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} x_1 = -4x_2 \\ x_1 + 4x_2 + 4x_3 = 0 \\ x_3 = 0 \end{matrix}$$

$$v_1 = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} t, \quad t \in \mathbb{R} \setminus \{0\}$$

$$\lambda_2 = 1$$

$$\begin{pmatrix} -3 & -8 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + 3x_2 + 4x_3 = 0 \rightarrow x_1 = -4x_3$$
$$x_2 = 0$$

$$v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \cdot t, \quad t \in \mathbb{R} \setminus \{0\}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 2$$

$$\begin{pmatrix} -4 & -8 & -12 \\ 1 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -x_3 &= 0 \\ x_1 + 2x_2 + 4x_3 &= 0 \\ x_1 &= -2x_2 \end{aligned}$$

$$v_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} t, \quad t \in \mathbb{R} \setminus \{0\}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ -1 & 1 & 1-\lambda & 1 \\ -1 & 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} =$$

$$(1-\lambda) [-\lambda(1-\lambda)(2-\lambda) + 1(1-\lambda)] = (1-\lambda)^2 [-\lambda(2-\lambda) + 1]$$

$$(1-\lambda)^2 (-2\lambda + \lambda^2 + 1) = (1-\lambda)^2 (1-\lambda)^2 = 0$$

$$\lambda = 1$$

$$\begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 + x_4 = 0$$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$V = \{ c_1 v_1 + c_2 v_2 + c_3 v_3, \quad c_1, c_2, c_3 \in \mathbb{R} \} \setminus \{ \vec{0} \}$$

$$A(c_1 v_1 + c_2 v_2 + c_3 v_3) = c_1 A v_1 + c_2 A v_2 + c_3 A v_3 =$$

$\downarrow$   
 $\vec{1}$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 1 \cdot \vec{1}$$