

Measure Theory - Exercises

X set, $\mathcal{A} = \mathcal{P}(X)$ σ -algebra is given by the power set

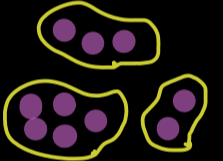
Counting measure: $\mu(A) := \begin{cases} \#A, & \text{if } A \text{ is a finite set} \\ \infty, & \text{if } A \text{ is not a finite set} \end{cases}$
number of elements

Show that this is a measure! First of all: $\mu: \mathcal{A} \rightarrow [0, \infty) \cup \{\infty\}$ ✓

(a) $\mu(\emptyset) = 0$ ✓

(b) A_1, A_2, A_3, \dots are subsets of X with $A_i \cap A_j = \emptyset$ for $i \neq j$

First case: All the sets are finite and $A_j = \emptyset$ for all $j \geq n$.

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} A_j\right) &= \mu\left(\bigcup_{j=1}^n A_j\right) = \#\bigcup_{j=1}^n A_j \stackrel{\text{disjoint}}{=} \sum_{j=1}^n \#A_j \\ &= \sum_{j=1}^n \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j) \end{aligned}$$


Second case: All the sets are finite and for each $n \in \mathbb{N}$ there is $j \geq n$ with $A_j \neq \emptyset$.

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} A_j\right) &= \infty \\ \sum_{j=1}^{\infty} \mu(A_j) &= \sum_{j=1}^{\infty} \#A_j = \infty \end{aligned}$$

infinitely many are ≥ 1

Third case: At least one of the sets is infinite: A_k

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} A_j\right) &= \infty \\ \sum_{j=1}^{\infty} \mu(A_j) &= \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \mu(A_j) + \underbrace{\mu(A_k)}_{= \infty} = \infty \end{aligned}$$

□