Measure Theory - Exercises

X set, $\mathcal{A} = \mathcal{P}(X)$ σ-algebra is given by the power set

Counting measure: $\mu(A) := \begin{cases} \underset{\nwarrow \text{ number of elements}}{\#A}, & \text{if } A \text{ is a finite set} \\ \infty, & \text{if } A \text{ is not a finite set} \end{cases}$

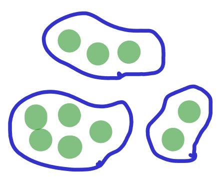
Show that this is a measure! First of all: $\mu : \mathcal{A} \longrightarrow [0,\infty) \cup \{\infty\}$ ✓

(a) $\mu(\emptyset) = 0$ ✓

(b) $A_1, A_2, A_3, \ldots$ are subsets of X with $A_i \cap A_j = \emptyset$ for $i \neq j$

First case: All the sets are finite and $A_j = \emptyset$ for all $j \geq n$.

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=1}^{n} A_j\right) = \#\bigcup_{j=1}^{n} A_j \overset{\text{disjoint}}{=} \sum_{j=1}^{n} \#A_j$$



$$= \sum_{j=1}^{n} \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j)$$

Second case: All the sets are finite and for each $n \in \mathbb{N}$ there is $j \geq n$ with $A_j \neq \emptyset$.

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \infty$$

$$\sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \underbrace{\#A_j}_{\text{infinitely many are } \geq 1} = \infty$$

Third case: At least one of the sets is infinite: A_k

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \infty$$

$$\sum_{j=1}^{\infty} \mu(A_j) = \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \mu(A_j) + \underbrace{\mu(A_k)}_{= \infty} = \infty$$

$\square$