BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Ordinary Differential Equations
Exercises 1

Exercise 1. Find the solutions of the following ODEs:

a) $x^2 y' = y^2$ c) $y' = (1 - y^2)$

b) $y'(1 + x^2) = x y$ d) $y' \sin y = -x$

a) $x^2 y' = y^2$

$y' = f(x) g(y)$ separation of variables

$y' = y^2 \frac{1}{x^2}$ $\frac{dy}{dx}\frac{1}{y^2} = \frac{1}{x^2}$, $\int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx$, $-\frac{1}{y} = -\frac{1}{x} + c$

$$y = \frac{x}{1 - cx} , \quad c \in \mathbb{R}$$

b) $y'(1 + x^2) = x y$

$y' = \frac{x}{(1+x^2)} y$

$\frac{dy}{dx} \frac{1}{y} = \frac{x}{1+x^2}$

$\int \frac{1}{y} dy = \frac{1}{2} \int \frac{2x}{1+x^2} dx$ $u = 1 + x^2$, $\frac{du}{dx} = 2x$

$\ln|y| = \frac{1}{2} \int \frac{1}{u} du$

$\ln|y| = \frac{1}{2} \ln(1+x^2) + c$

$e^{\ln|y|} = e^{\ln(1+x^2)^{1/2} + c}$

$\pm y = (1+x^2)^{1/2} \cdot e^c$

$y = \sqrt{1+x^2} \cdot \underbrace{(\pm e^c)}_{\hat{c}}$

$$y = \sqrt{1+x^2}\, \hat{c} , \quad \hat{c} \in \mathbb{R}$$

c) $y' = 1 - y^2$

$y' = f(x) g(y)$

$\frac{dy}{dx} \frac{1}{1-y^2} = 1$

$\int \frac{1}{1-y^2} dy = \int 1 \, dx$

$\ln \frac{|y+1|}{|y-1|} = (x + c)2$

$\frac{|y+1|}{|y-1|} = e^{2x+c}$

$\frac{d}{dx} \operatorname{artanh}(x) = \frac{1}{1-x^2}$, $|x| < 1$

$\frac{d}{dx} \operatorname{arcotanh}(x) = \frac{1}{1-x^2}$, $|x| > 1$

$\frac{1}{1-y^2} = \frac{A}{(y-1)} + \frac{B}{(y+1)}$

$\cdot (y-1)$ $\frac{(y-1)}{-(1-y)(1+y)} = A + \frac{(y-1)B}{(y+1)} \xrightarrow{y=1} -\frac{1}{2} = A$

$\cdot (y+1)$ $\frac{1}{1-y} = -\frac{1}{2}\frac{1}{(y-1)}(y+1) + B \xrightarrow{y=-1} \frac{1}{2} = B$

$\int \frac{1}{1-y^2} dy = -\frac{1}{2} \int \frac{1}{y-1} dy + \frac{1}{2} \int \frac{1}{y+1} dy = -\frac{1}{2} \ln|y-1| + \frac{1}{2} \ln|y+1| = \frac{1}{2} \ln \frac{|y+1|}{|y-1|}$

① $|y| > 1$, $\frac{y+1}{y-1} = e^{2x+c}$

② $|y| < 1$, $\frac{y+1}{1-y} = e^{2x+c}$

① $y + 1 = y e^{2x+c} - e^{2x+c}$

$y(1 - e^{2x+c}) = -1 - e^{2x+c}$

$y = \frac{1 + e^{2x+c}}{e^{2x+c} - 1} \cdot \frac{e^{-x-\frac{c}{2}}}{e^{-x-\frac{c}{2}}}$

$y = \frac{e^{-(x+\frac{c}{2})} + e^{x+\frac{c}{2}}}{e^{x+\frac{c}{2}} - e^{-(x+\frac{c}{2})}}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$

$$y = \operatorname{cotanh}(x + \hat{c}) , \quad |y| > 1 , \quad \hat{c} \in \mathbb{R}$$

② $y + 1 = e^{2x+c} - y e^{2x+c}$

$y(1 + e^{2x+c}) = e^{2x+c} - 1$

$y = \frac{e^{x+\frac{c}{2}} - e^{-(x+\frac{c}{2})}}{e^{-(x+\frac{c}{2})} + e^{x+\frac{c}{2}}}$

$$y = \tanh(x + \hat{c}) , \quad |y| < 1 , \quad \hat{c} \in \mathbb{R}$$

d) $y' \sin y = -x$

$y' = -x \cdot \frac{1}{\sin y}$

$\frac{dy}{dx} \sin y = -x$

$\int \sin y \, dy = -\int x \, dx$

$-\cos y = -\frac{x^2}{2} + C$

$$y = \arccos\left(\frac{x^2}{2} - c\right) , \quad c \in \mathbb{R}$$