The Bright Side of
Mathematics

Limits of sequences and series

Exercise 1. Find the limits:

a) $\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{2}n+2}$ b) $\lim_{n\to\infty}\frac{\sqrt{\frac{n-1}{n}}-1}{\frac{n-1}{n}-1}$

Small hint for a) : $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\lim_{n\to\infty} a_n = a \quad \lim_{n\to\infty} b_n = b$

- $\lim(a_n+b_n) = \lim a_n + \lim b_n$
- $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n$

$\lim\left(\frac{a_n}{b_n}\right)$ choose $b_n' = \frac{1}{b_n}$ $\lim b_n' = \frac{1}{b}$

$\lim(a_n - b_n)$ choose $b_n' = -b_n$ $\lim b_n' = -b$

a) $\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{2}n+2}$

$= \lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{2}n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^2$

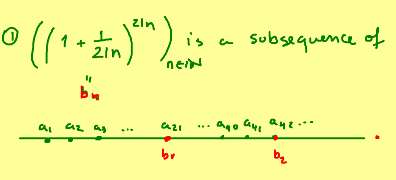
$= \lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{21}\cdot\frac{1}{2}\cdot 21n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^2$

$= \lim_{n\to\infty}\left(\left(1+\frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}}\cdot\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^2 = \boxed{e^{\frac{1}{42}}\cdot 1}$

Def. We say $\lim_{n\to\infty} a_n = a$, $(a_n)_{n\in\mathbb{N}}$ real sequence, when for every $\varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t. $|a_n - a| < \varepsilon$ for $n > N$.

① $\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^{21n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

① $\left(\left(1+\frac{1}{21n}\right)^{21n}\right)_{n\in\mathbb{N}}$ is a subsequence of $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$
(b_n for the first, a_n for the second)



If $a_n \to a$, then $b_n \to a$

② $\lim_{n\to\infty}\left(\left(1+\frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} = \left(\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} = e^{\frac{1}{42}}$

$\lim_{n\to\infty}\left(1+\frac{1}{21n}\right)^2 = 1$

② Say $\lim a_n = a$. Is it true that $\lim a_n^{\frac{1}{42}} = (\lim a_n)^{\frac{1}{42}} = a^{\frac{1}{42}}$? $a_n > 1$ and $a > 1$

Given $\varepsilon>0$: $|a_n - a| < \varepsilon$ for $n \geq N$. What happens to $|a_n^{\frac{1}{42}} - a^{\frac{1}{42}}|$?

We want something like:
$\varepsilon > |a_n - a| > |a_n^{\frac{1}{42}} - a^{\frac{1}{42}}|$

We use $(x^{42} - y^{42}) = (x-y)(x^{41}y^0 + x^{40}y^1 + \dots + x^1y^{40} + x^0y^{41})$ $\quad x = a^{\frac{1}{42}} \quad y = a_n^{\frac{1}{42}}$

$(x^{42} + x^{41}y + \dots + x^2y^{40} + xy^{41}$
$- x^{41}y^1 - x^{40}y^2 - \dots - x^1y^{41} - y^{42})$

$(a - a_n) = (a^{\frac{1}{42}} - a_n^{\frac{1}{42}})(a^{\frac{41}{42}} + a^{\frac{40}{42}}a_n^{\frac{1}{42}} + \dots)$

$|a - a_n| = |a^{\frac{1}{42}} - a_n^{\frac{1}{42}}|\,\underbrace{|a^{\frac{41}{42}} + a^{\frac{40}{42}}a_n^{\frac{1}{42}} + \dots + a_n^{\frac{41}{42}}|}_{>1}$ $\Rightarrow \varepsilon > |a - a_n| > |a^{\frac{1}{42}} - a_n^{\frac{1}{42}}|$ for $n > N$.