Limits of sequences and series

Exercise 1. Find the limits:

a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{2}n+2}$ b) $\lim\limits_{n\to\infty}\dfrac{\sqrt{\frac{n-1}{n}}-1}{\frac{n-1}{n}-1}$

Small hint for a) : $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{2}n+2}$

$= \lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{2}n}\cdot\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{2}$

$= \lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{\frac{1}{21}\cdot\frac{1}{2}\cdot 21n}\cdot\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{2}$

$= \lim\limits_{n\to\infty}\left(\left(1+\frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}}\cdot\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{2} = \boxed{e^{\frac{1}{42}}\cdot 1}$

① $\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{21n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$

② $\lim\limits_{n\to\infty}\left(\left(1+\frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} = \left(\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} = e^{\frac{1}{42}}$

$\lim\limits_{n\to\infty}\left(1+\frac{1}{21n}\right)^{2} = 1$

$\lim\limits_{n\to\infty} a_n = a$ $\lim\limits_{n\to\infty} b_n = b$

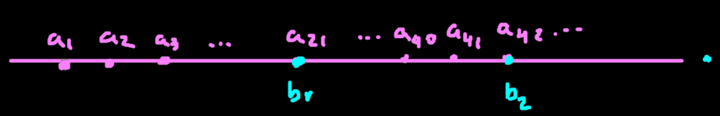
- $\lim(a_n+b_n) = \lim a_n + \lim b_n$
- $\lim(a_n\cdot b_n) = \lim a_n\cdot\lim b_n$

$\lim\left(\frac{a_n}{b_n}\right)$ choose $b_n' = \frac{1}{b_n}$ $\lim b_n' = \frac{1}{b}$

$\lim(a_n-b_n)$ choose $b_n' = -b_n$ $\lim b_n' = -b$

Def. We say $\lim\limits_{n\to\infty} a_n = a$, $(a_n)_{n\in\mathbb{N}}$ real sequence, when for every $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $|a_n-a|<\varepsilon$ for $n>N$.

① $\underbrace{\left(\left(1+\frac{1}{21n}\right)^{21n}\right)_{n\in\mathbb{N}}}_{b_n}$ is a subsequence of $\underbrace{\left(\left(1+\frac{1}{n}\right)^{n}\right)_{n\in\mathbb{N}}}_{a_n}$



If $a_n\to a$, then $b_n\to a$

② Say $\lim a_n = a$. Is it true that $\lim a_n^{\frac{1}{42}} = (\lim a_n)^{\frac{1}{42}} = a^{\frac{1}{42}}$? $a_n>1$ and $a>1$

Given $|a_n-a|<\varepsilon$ for $n\geq N$. What happens to $|a_n^{\frac{1}{42}}-a^{\frac{1}{42}}|$?
$\varepsilon>0$

We want something like:
$\varepsilon > |a_n-a| > |a_n^{\frac{1}{42}}-a^{\frac{1}{42}}|$

We use $(x^{42}-y^{42}) = (x-y)(x^{41}y^0+x^{40}y^1+\dots+x^1y^{40}+x^0y^{41})$ $x = a^{\frac{1}{42}}$ $y = a_n^{\frac{1}{42}}$

$(x^{42}+\cancel{x^{41}y}+\cancel{\dots}+\cancel{x^2y^{40}}+\cancel{xy^{41}} - \cancel{x^{41}y^1}-\cancel{x^{40}y^2}-\cancel{\dots}-\cancel{x^1y^{41}}-y^{42})$

$(a-a_n) = (a^{\frac{1}{42}}-a_n^{\frac{1}{42}})(a^{\frac{41}{42}}+a^{\frac{40}{42}}a_n^{\frac{1}{42}}+\dots)$

$|a-a_n| = |a^{\frac{1}{42}}-a_n^{\frac{1}{42}}|\,|\underbrace{a^{\frac{41}{42}}+a^{\frac{40}{42}}a_n^{\frac{1}{42}}+\dots+a_n^{\frac{41}{42}}}_{>1}|$ $\Rightarrow \varepsilon>|a-a_n|>|a^{\frac{1}{42}}-a_n^{\frac{1}{42}}|$ for $n>N$.