



Limits of sequences and series

Exercise 1. Find the limits:

a) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^{\frac{1}{2}n+2}$

b) $\lim_{n \rightarrow \infty} \frac{\sqrt{\frac{n-1}{n}} - 1}{\frac{n-1}{n} - 1}$

Small hint for a): $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

$\lim_{n \rightarrow \infty} a_n = a \quad \lim_{n \rightarrow \infty} b_n = b$

• $\lim (a_n + b_n) = \lim a_n + \lim b_n$

• $\lim (a_n \cdot b_n) = \lim a_n \cdot \lim b_n$

$\lim \left(\frac{a_n}{b_n}\right)$ choose $b_n' = \frac{1}{b_n} \quad \lim b_n' = \frac{1}{b}$

$\lim (a_n - b_n)$ choose $b_n' = -b_n \quad \lim b_n' = -b$

Def. We say $\lim_{n \rightarrow \infty} a_n = a$, $(a_n)_{n \in \mathbb{N}}$ real sequence,

when for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$|a_n - a| < \epsilon$ for $n > N$.

a) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^{\frac{1}{2}n+2}$

$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^{\frac{1}{2}n} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^2$

$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^{\frac{1}{21} \cdot \frac{1}{2} \cdot 21n} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^2$

$= \lim_{n \rightarrow \infty} \left(\left(1 + \frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^2 = \boxed{e^{\frac{1}{42}} \cdot 1}$

① $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^{21n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

② $\lim_{n \rightarrow \infty} \left(\left(1 + \frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} = \left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^{21n}\right)^{\frac{1}{42}} = e^{\frac{1}{42}}$

$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{21n}\right)^2 = 1$

② Say $\lim a_n = a$. Is it true that $\lim a_n^{\frac{1}{42}} = (\lim a_n)^{\frac{1}{42}} = a^{\frac{1}{42}}$? $a_n > 1$ and $a > 1$

Given $|a_n - a| < \epsilon$ for $n \geq N$. What happens to $|a_n^{\frac{1}{42}} - a^{\frac{1}{42}}|$?

We want something like:

$\epsilon > |a_n - a| > |a_n^{\frac{1}{42}} - a^{\frac{1}{42}}|$

We use $(x^{42} - y^{42}) = (x - y)(x^{41}y^0 + x^{40}y^1 + \dots + x^1y^{40} + x^0y^{41})$

$x = a^{\frac{1}{42}} \quad y = a_n^{\frac{1}{42}}$

$(x^{42} + x^{41}y + \dots + x^2y^{40} + xy^{41} - x^{41}y - x^{40}y^2 - \dots - xy^{41} - y^{42})$

$(a - a_n) = (a^{\frac{1}{42}} - a_n^{\frac{1}{42}})(a^{\frac{41}{42}} + a^{\frac{40}{42}}a_n^{\frac{1}{42}} + \dots)$

$|a - a_n| = |a^{\frac{1}{42}} - a_n^{\frac{1}{42}}| \underbrace{\left| a^{\frac{41}{42}} + a^{\frac{40}{42}}a_n^{\frac{1}{42}} + \dots + a_n^{\frac{41}{42}} \right|}_{> 1} \Rightarrow \epsilon > |a - a_n| > |a^{\frac{1}{42}} - a_n^{\frac{1}{42}}| \text{ for } n > N.$

① $\left(\left(1 + \frac{1}{21n}\right)^{21n}\right)_{n \in \mathbb{N}}$ is a subsequence of $\left(\left(1 + \frac{1}{n}\right)^n\right)_{n \in \mathbb{N}}$



If $a_n \rightarrow a$, then $b_n \rightarrow a$