The Bright Side of Mathematics

The following pages cover the whole Fourier Transform course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Fourier Transform – Part 1

↳ applications in physics, computer science,...

(JPEG compression, equalization of audio recordings,...)

Fourier series

$f: \mathbb{R} \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

periodic function

$f: [a,b) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

↑↓ transform

$\hat{f}: \mathbb{Z} \longrightarrow \mathbb{C}$

continuous Fourier transform

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

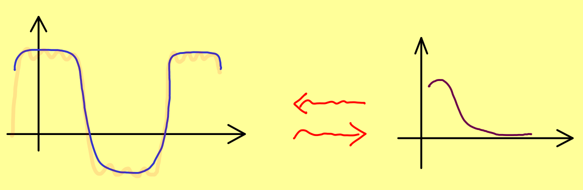
↑↓ transform

$\hat{f}: \mathbb{R}^n \longrightarrow \mathbb{C}$

(**Fourier analysis on groups**

general measure theory)

Idea of Fourier transform:



time domain ⇄ frequency domain

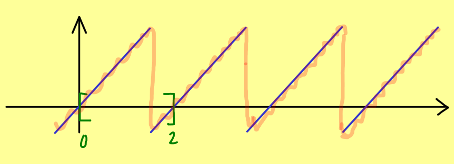
Requirements:

Real Analysis ⇒ Fourier Transform

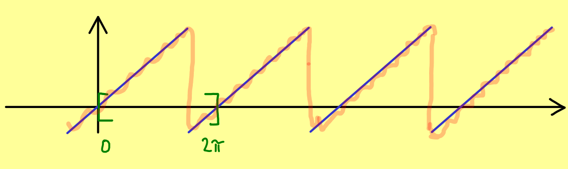
Linear Algebra, Abstract Linear Algebra ⇒ Fourier Transform

Fourier Transform – Part 2

Idea of Fourier series:



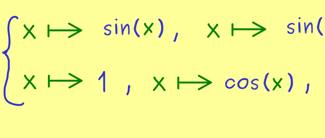
The function is 2-periodic: $f(x+2) = f(x)$ for all $x \in \mathbb{R}$

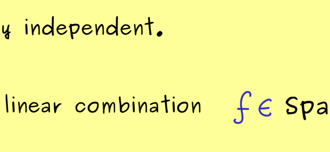


$$\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R} \}$$

→ real vector space

Example:

- constant function $f(x) = 5$
- 

- 

$x \mapsto \sin(x)$
$x \mapsto \sin(2x)$

Proposition: $\mathcal{U} \subseteq \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$ given by

$$\mathcal{U} := \left\{ \begin{array}{l} x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \dots, \\ x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \dots \end{array} \right\}$$

(odd functions: first row; even functions: second row)

is linearly independent.

Definition: A linear combination $f \in \operatorname{Span}(\mathcal{U})$, $f : \mathbb{R} \to \mathbb{R}$, is called (real) trigonometric polynomial:

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x), \quad a_i, b_i \in \mathbb{R}$$

For $\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$, we have a (complex) trigonometric polynomial:

$$f(x) = \sum_{k=-n}^{n} c_k \cdot \exp(i \cdot k \cdot x), \quad c_k \in \mathbb{C}$$

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Fourier Transform – Part 3

In $\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$, we have (real) trigonometric polynomials:

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) \quad , \quad a_i, b_i \in \mathbb{R}$$

Subspace: $$\mathcal{P}_{2\pi\text{-per}} := \operatorname{Span}\big(\underbrace{x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \dots,\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \dots}_{\text{basis!}}\big)$$

Definition: For $f, g \in \mathcal{P}_{2\pi\text{-per}}$, we define an inner product:

$$\langle f, g \rangle := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

Example:

$$\langle x \mapsto 1, x \mapsto 1 \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} 1 \, dx = 1$$

$$\langle x \mapsto \cos(x), x \mapsto \sin(x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(x) \sin(x)\, dx$$

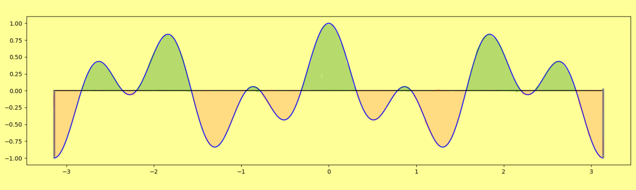
$$= \frac{1}{2\pi} \left(\frac{1}{2} (\sin(x))^2 \Big|_{-\pi}^{\pi} \right) = 0$$

$$\langle x \mapsto \cos(k \cdot x), x \mapsto \sin(m \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\cos(k \cdot x) \sin(m \cdot x)}_{\text{odd function}}\, dx = 0$$

$$\langle x \mapsto 1, x \mapsto \cos(k \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k \cdot x)\, dx = \frac{1}{2\pi} \frac{1}{k} \sin(k \cdot x) \Big|_{-\pi}^{\pi} = 0$$

$$\langle x \mapsto 1, x \mapsto \sin(m \cdot x) \rangle = 0$$

$$\langle x \mapsto \cos(k \cdot x), x \mapsto \cos(m \cdot x) \rangle = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(k \cdot x) \cos(m \cdot x)\, dx$$



$$= 0 \quad \text{if } k \neq m$$

Use: $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

Then: $$\int_{-\pi}^{\pi} \cos(k \cdot x) \cos(m \cdot x)\, dx = \frac{1}{4} \int_{-\pi}^{\pi} \left(e^{i(k+m)x} + e^{-i(k+m)x} + e^{i(k-m)x} + e^{-i(k-m)x} \right) dx$$

$k \neq m$

$$= \frac{1}{4} \left(\frac{1}{i(k+m)} e^{i(k+m)x} + \frac{1}{-i(k+m)} e^{-i(k+m)x} + \frac{1}{i(k-m)} e^{i(k-m)x} + \frac{1}{-i(k-m)} e^{-i(k-m)x} \right) \Big|_{-\pi}^{\pi}$$

Use: $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$$= \frac{1}{2} \left(\frac{1}{k+m} \sin((k+m) \cdot x) + \frac{1}{k+m} \sin((k-m) \cdot x) \right) \Big|_{-\pi}^{\pi} = 0$$

And similarly: $$\int_{-\pi}^{\pi} \sin(k \cdot x) \sin(m \cdot x)\, dx \overset{k \neq m}{=} 0$$

Result: $$\mathcal{B} = \big(x \mapsto 1,\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \dots,\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \dots \big)$$

satisfies $\langle f, g \rangle = 0 \quad f \neq g, \ f, g \in \mathcal{B}$

$\rightsquigarrow$ $\mathcal{B}$ orthogonal basis (OB)

$\rightsquigarrow$ make to orthonormal basis (ONB)

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Fourier Transform – Part 4

We already know: we have an orthogonal basis (OB)

$$\mathcal{B} = \big(x \mapsto 1 ,\ x \mapsto \cos(x) ,\ x \mapsto \cos(2x) ,\ x \mapsto \cos(3x) , \dots ,\ x \mapsto \sin(x) ,\ x \mapsto \sin(2x) ,\ x \mapsto \sin(3x) , \dots \big)$$

for $\mathcal{P}_{2\pi\text{-per}}$ with inner product $\langle f, g \rangle_1 := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

Normalize: $\langle x \mapsto \sin(kx) , x \mapsto \sin(kx) \rangle_1 = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} (\sin(kx))^2 \, dx$

$$\int_{-\pi}^{\pi} (\sin(kx))^2 \, dx = \int_{-\pi}^{\pi} \underbrace{\sin(kx)}_{u} \underbrace{\sin(kx)}_{v'} \, dx = \sin(kx)\left(-\tfrac{1}{k}\right)\cos(kx) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} k\cos(kx)\left(-\tfrac{1}{k}\right)\cos(kx)\, dx$$

integration by parts: $u' = k\cos(kx)$, $v = -\frac{1}{k}\cos(kx)$

$$= \int_{-\pi}^{\pi} \underbrace{(\cos(kx))^2}_{1-(\sin(kx))^2} \, dx$$

$$\Longrightarrow \quad 2 \cdot \int_{-\pi}^{\pi} (\sin(kx))^2 \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$\langle x \mapsto \sin(kx) , x \mapsto \sin(kx) \rangle_1 = \frac{1}{2} \quad \rightsquigarrow \quad \text{length} = \frac{1}{\sqrt{2}}$$

Hence: $x \mapsto \sqrt{2} \cdot \sin(kx)$ has norm 1

Proposition:

(1) $$\mathcal{B} = \big(x \mapsto 1 ,\ x \mapsto \sqrt{2}\cos(x),\ x \mapsto \sqrt{2}\cos(2x) ,\ x \mapsto \sqrt{2}\cos(3x) , \dots ,\ x \mapsto \sqrt{2}\sin(x),\ x \mapsto \sqrt{2}\sin(2x),\ x \mapsto \sqrt{2}\sin(3x) , \dots \big)$$

is an ONB w.r.t. the inner product: $\langle f, g \rangle_1 := \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

(2) $$\mathcal{B} = \Big(x \mapsto \tfrac{1}{\sqrt{2\pi}} ,\ x \mapsto \tfrac{1}{\sqrt{\pi}}\cos(x) ,\ x \mapsto \tfrac{1}{\sqrt{\pi}}\cos(2x) ,\ x \mapsto \tfrac{1}{\sqrt{\pi}}\cos(3x) , \dots ,\ x \mapsto \tfrac{1}{\sqrt{\pi}}\sin(x),\ x \mapsto \tfrac{1}{\sqrt{\pi}}\sin(2x),\ x \mapsto \tfrac{1}{\sqrt{\pi}}\sin(3x) , \dots \Big)$$

is an ONB w.r.t. the inner product: $\langle f, g \rangle_2 := \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

(3) $$\mathcal{B} = \Big(x \mapsto \tfrac{1}{\sqrt{2}} ,\ x \mapsto \cos(x) ,\ x \mapsto \cos(2x) ,\ x \mapsto \cos(3x) , \dots ,\ x \mapsto \sin(x) ,\ x \mapsto \sin(2x) ,\ x \mapsto \sin(3x) , \dots \Big)$$

is an ONB w.r.t. the inner product: $\langle f, g \rangle_3 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

For trigonometric polynomials:

$$f(x) = \tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) \quad , \quad a_i, b_i \in \mathbb{R}$$

Fourier coefficients w.r.t. ONB in (3)

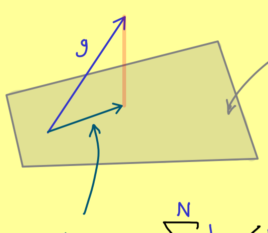
$$a_k = \langle x \mapsto \cos(k \cdot x) , f \rangle_3 \quad , \quad \tilde{a}_0 = \langle x \mapsto \tfrac{1}{\sqrt{2}} , f \rangle_3$$

$$b_k = \langle x \mapsto \sin(k \cdot x) , f \rangle_3$$

Approximation of periodic functions?

$g : \mathbb{R} \to \mathbb{R}$

2π-periodic + "integrable"



trigonometric polynomials with basis:

$\mathcal{B} = (h_1, h_2, \dots, h_N)$

ONB!

orthogonal projection $= \sum_{k=1}^{N} h_k \langle h_k , g \rangle$

Fourier Transform – Part 5

$$\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) = \{ f: \mathbb{R} \to \mathbb{C} \mid f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R} \}$$

$$\mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \operatorname{Span}\Big(x \mapsto \tfrac{1}{\sqrt{2}},\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \dots,\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \dots \Big)$$

↳ inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx$

Let's take integrable functions:

$$\mathcal{L}^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) = \{ f \in \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \mid \int_{-\pi}^{\pi} |f(x)|\, dx < \infty \}$$

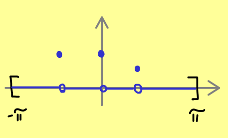
↳ complex vector space

f integrable with respect to Lebesgue measure on $[-\pi, \pi]$

norm? $\|f\|_1 := \int_{-\pi}^{\pi} |f(x)|\, dx$

↳ not a norm on $\mathcal{L}^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

problem:



solution: equivalence relation $f \sim g :\Leftrightarrow \|f-g\|_1 = 0$

set of all equivalence classes: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \mathcal{L}^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) / \sim$

↳ complex vector space

$\|[f]\|_1 := \|f\|_1$

↳ norm!

identify: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

Let's take square-integrable functions:

$$\mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) = \{ f \in \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \mid \int_{-\pi}^{\pi} |f(x)|^2\, dx < \infty \}$$

norm? $\|f\|_2 := \sqrt{\int_{-\pi}^{\pi} |f(x)|^2\, dx}$

solution: equivalence relation $f \sim g :\Leftrightarrow \|f-g\|_2 = 0$

set of all equivalence classes: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) / \sim$

↳ complex vector space with inner product

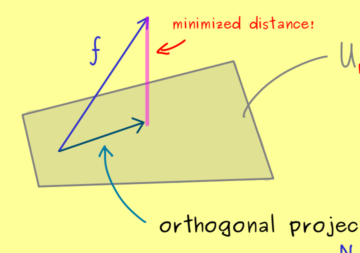
Fourier Transform – Part 6

We know: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

inner product: $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x)\, dx$

Orthogonality: $\mathcal{B}_n = \left(x \mapsto \frac{1}{\sqrt{2}},\ x \mapsto \cos(x),\ x \mapsto \cos(2x), \ldots, x \mapsto \cos(nx),\ x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \ldots, x \mapsto \sin(nx) \right)$

ONS in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ for every $n \in \mathbb{N}$



U_n finite-dimensional subspace spanned by $\mathcal{B}_n$

write: $\mathcal{B}_n = (h_1, h_2, \ldots, h_N)$, $N = 2n+1$

orthogonal projection of f onto U_n:

$$\mathcal{F}_n(f) = \sum_{k=1}^{N} h_k \underbrace{\langle h_k, f \rangle}_{\text{Fourier coefficients}}$$

Definition:

$$\mathcal{F}_n(f)(x) = \tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x)$$

with $\tilde{a}_0 = \left\langle x \mapsto \frac{1}{\sqrt{2}}, f \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\, dx$

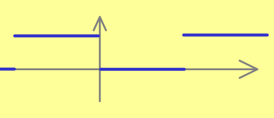
$$a_k = \langle x \mapsto \cos(k \cdot x), f \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k \cdot x) f(x)\, dx$$

$$b_k = \langle x \mapsto \sin(k \cdot x), f \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(k \cdot x) f(x)\, dx$$

The map $h \mapsto \mathcal{F}_n(f)(x)$ (with $x \in \mathbb{R}$)

is called the Fourier series of $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ (can be extended to $f \in L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$)

Example: $f: \mathbb{R} \to \mathbb{C}$, $f(x) = \begin{cases} 1, & x \in (-\pi, 0) \\ 0, & x \in [0, \pi] \end{cases}$



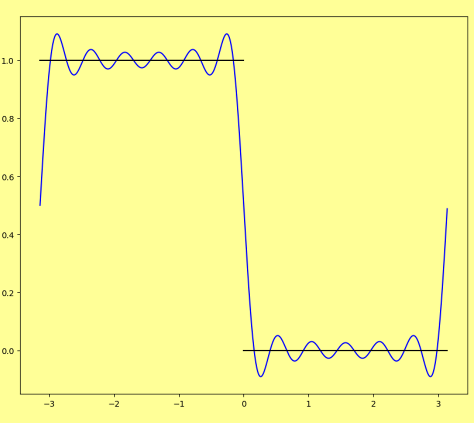
$$\tilde{a}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k \cdot x) f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} \cos(k \cdot x)\, dx = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(k \cdot x) f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} \sin(k \cdot x)\, dx = \frac{1}{\pi}\left(-\frac{1}{k}\cos(k \cdot x) \right)\Big|_{-\pi}^{0}$$

$$= \begin{cases} 0, & k \text{ even} \\ -\frac{2}{\pi k}, & k \text{ odd} \end{cases}$$

Fourier series: $\frac{1}{2} + \frac{-2}{\pi}\sin(x) + \frac{-2}{\pi 3} \cdot \sin(3 \cdot x) + \frac{-2}{\pi 5} \cdot \sin(5 \cdot x) + \ldots$



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Fourier Transform – Part 7

$f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \xrightarrow{\text{orthogonal projection}} \mathcal{F}_n(f)$

→ trigonometric polynomial
(cos and sin functions)
→ exponential functions

Euler's formula:

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Example:

$$A \cdot \cos(x) + B \cdot \cos(2x) + C \sin(2x), \qquad A, B, C \in \mathbb{C}$$

$$= \frac{A}{2}\left(e^{ix} + e^{-ix}\right) + \frac{B}{2}\left(e^{i2x} + e^{-i2x}\right) + \frac{C}{2i}\left(e^{i2x} - e^{-i2x}\right)$$

$$= \frac{A}{2} \cdot e^{ix} + \frac{A}{2} \cdot e^{-ix} + \left(\frac{B}{2} + \frac{C}{2i}\right) e^{i2x} + \left(\frac{B}{2} - \frac{C}{2i}\right) e^{-i2x}$$

→ complex linear combination!

Remember: In $\mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$:

$$\text{Span}\Big(x \mapsto \tfrac{1}{\sqrt{2}},\ x \mapsto \cos(x),\ x \mapsto \cos(2x), \ldots, x \mapsto \cos(nx),$$
$$x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \ldots, x \mapsto \sin(nx)\Big)$$

$$= \text{Span}\left(x \mapsto e^{-inx}, \ldots, x \mapsto e^{-ix}, x \mapsto e^{i0\cdot x}, x \mapsto e^{ix}, \ldots, x \mapsto e^{inx}\right)$$

and

$$\tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cdot \cos(k \cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k \cdot x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

with

$$c_k = \begin{cases} \frac{1}{2}\left(a_k + \frac{b_k}{i}\right), & \text{for } k > 0 \\ \tilde{a}_0 \frac{1}{\sqrt{2}} & \text{for } k = 0 \\ \frac{1}{2}\left(a_{-k} - \frac{b_{-k}}{i}\right), & \text{for } k < 0 \end{cases}$$

Result: Take $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

with inner product:

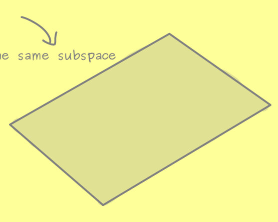
$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x)\, dx$$

↑ best factor for exponential functions

ONS: $\mathcal{B}_n = \Big(x \mapsto 1,\ x \mapsto \sqrt{2}\cos(x),\ x \mapsto \sqrt{2}\cos(2x),\ x \mapsto \sqrt{2}\cos(3x), \ldots, x \mapsto \sqrt{2}\cos(nx),$
$x \mapsto \sqrt{2}\sin(x),\ x \mapsto \sqrt{2}\sin(2x),\ x \mapsto \sqrt{2}\sin(3x), \ldots, x \mapsto \sqrt{2}\sin(nx)\Big)$

ONS: $\mathcal{E}_n = \left(x \mapsto e^{ikx}\right)_{k=-n,\ldots,n} = \left(e_k\right)_{k=-n,\ldots,n}$

they span the same subspace



For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$: $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \underbrace{\langle e_k, f \rangle}_{\text{Fourier coefficients}}$

$$\Rightarrow \mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} c_k e^{ikx}, \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$$

The map $n \mapsto \mathcal{F}_n(f)$ is called the Fourier series of $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

(with complex coefficients)

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Fourier Transform – Part 8

Fourier series: $f \in L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \rightsquigarrow \mathcal{F}_n(f) \in \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

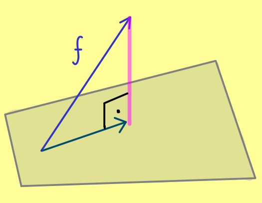
trigonometric polynomial

$$\mathcal{F}_n(f) = \sum_{k=-n}^{n} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$$

Geometric picture: For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \rightsquigarrow \mathcal{F}_n(f) \in \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

$\mathcal{F}_n(f)$ → orthogonal projection



$\mathcal{F}_n(f) \perp \underbrace{f - \mathcal{F}_n(f)}_{\text{normal component}}$

Question: What happens for $n \to \infty$? $\mathcal{F}_n(f) \xrightarrow{n\to\infty} f$?

Proposition: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with inner product $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x)\, dx$

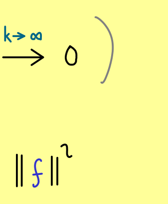
and ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k : x \mapsto e^{ikx}$.

Then for $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ and $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \underbrace{\langle e_k, f \rangle}_{c_k}$,

we have:

(a) $\| f - \mathcal{F}_n(f) \|^2 = \| f \|^2 - \sum_{k=-n}^{n} |c_k|^2$

L^2-norm $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$



Pythagorean theorem: $\| f \|^2 = \| \mathcal{F}_n(f) \|^2 + \| f - \mathcal{F}_n(f) \|^2$

(b) $\sum_{k=-n}^{n} |c_k|^2 \leq \| f \|^2$ for all n (Bessel's inequality)

$\left(\Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 \leq \| f \|^2 \quad \text{and} \quad c_k \xrightarrow{k\to\infty} 0 \right)$

(c) $\| f - \mathcal{F}_n(f) \| \xrightarrow{n\to\infty} 0 \iff \sum_{k=-\infty}^{\infty} |c_k|^2 = \| f \|^2$

(Parseval's identity)

Fourier Transform – Part 9

$L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ has ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k : x \mapsto e^{ikx}$

$\rightsquigarrow$ Fourier series $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \langle e_k, f\rangle$

Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f\rangle|^2$

$$\iff \|f - \mathcal{F}_n(f)\| \xrightarrow{n\to\infty} 0$$

means: $f = \mathcal{F}_n(f) + r_n$ with $\|r_n\| \xrightarrow{n\to\infty} 0$

Consider two functions: $f, g \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

$\langle f, g\rangle \leftarrow$ formula with Fourier coefficients?

$f = \mathcal{F}_n(f) + r_n$ with $\|r_n\| \xrightarrow{n\to\infty} 0$

$g = \mathcal{F}_n(g) + \tilde{r}_n$ with $\|\tilde{r}_n\| \xrightarrow{n\to\infty} 0$

We have: $|\langle \mathcal{F}_n(g), r_n\rangle| \underset{\text{Cauchy Schwarz}}{\leq} \|\mathcal{F}_n(g)\|\,\|r_n\|$

$\leq \|g\|\cdot\|r_n\| \xrightarrow{n\to\infty} 0$ (Bessel's inequality)

$$\begin{aligned}\langle f, g\rangle &= \langle \mathcal{F}_n(f) + r_n, \mathcal{F}_n(g) + \tilde{r}_n\rangle \\ &= \langle \mathcal{F}_n(f), \mathcal{F}_n(g)\rangle + \underbrace{\langle r_n, \mathcal{F}_n(g)\rangle + \langle \mathcal{F}_n(f), \tilde{r}_n\rangle + \langle r_n, \tilde{r}_n\rangle}_{(*)} \\ &= \Big\langle \sum_{k=-n}^{n} e_k \langle e_k, f\rangle, \sum_{\ell=-n}^{n} e_\ell \langle e_\ell, g\rangle \Big\rangle + (*) \\ &= \sum_{k=-n}^{n}\sum_{\ell=-n}^{n} \overline{\langle e_k, f\rangle}\langle e_\ell, g\rangle \underbrace{\langle e_k, e_\ell\rangle}_{=\delta_{k\ell}} + (*) \\ &= \sum_{k=-n}^{n} \langle f, e_k\rangle\langle e_k, g\rangle + (*) \\ &\xrightarrow{n\to\infty} \sum_{k=-\infty}^{\infty} \langle f, e_k\rangle\langle e_k, g\rangle\end{aligned}$$

Remember the equivalent statements: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with ONS $(e_k)_{k\in\mathbb{Z}}$

(a) Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f\rangle|^2$

(b) ONS is complete: $\|f - \sum_{k=-n}^{n} e_k\langle e_k, f\rangle\| \xrightarrow{n\to\infty} 0$

$\left(f = \sum_{k=-\infty}^{\infty} e_k \langle e_k, f\rangle \right)$

(c) ONS gives inner product:

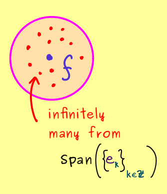
$\langle f, g\rangle = \sum_{k=-\infty}^{\infty} \langle f, e_k\rangle\langle e_k, g\rangle$

informal: $\left(\sum_{k=-\infty}^{\infty} |e_k\rangle\langle e_k| = \mathbb{1} \right)$

(d) ONS is total: $\text{Span}\left(\{e_k\}_{k\in\mathbb{Z}}\right)$ is dense in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$:

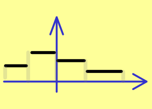
$\forall f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})\ \forall \varepsilon > 0\ \exists N \in \mathbb{N},\ \lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{C}$:

$$\Big\| f - \sum_{k=-N}^{N} \lambda_k e_k \Big\| < \varepsilon$$



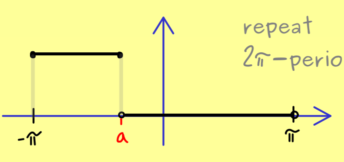
Fourier Transform – Part 10

For proving Parseval's identity ⟿ step functions



Most important step function:

$$h_a(x) = \begin{cases} 1, & x \in [-\pi, a] \\ 0, & x \in (a, \pi) \end{cases}$$



repeat 2π-periodically

for every $a \in [-\pi, \pi]$

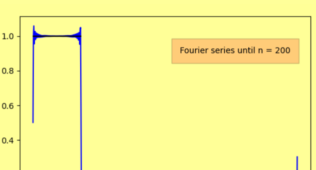
Fourier series for this example:

$$c_k = \langle e_k, h_a \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} h_a(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{a} e^{-ikx}\, dx$$

$$= \begin{cases} \frac{a + \pi}{2\pi}, & k = 0 \\ \frac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right), & k \neq 0 \end{cases}$$

Visualization:

$$a_k = 2 \cdot \mathrm{Re}(c_k)$$
$$b_k = -2 \cdot \mathrm{Im}(c_k)$$



Show Parseval's identity:

$k \neq 0$: $$|c_k|^2 = \frac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right) \overline{\frac{1}{2\pi(-ik)} \left(e^{-ika} - e^{ik\pi} \right)}$$

$$= \frac{1}{4\pi^2 k^2} \cdot \left(e^{-ika} - e^{ik\pi} \right) \cdot \left(e^{ika} - e^{-ik\pi} \right)$$

$$= \frac{1}{4\pi^2 k^2} \cdot \left(1 - e^{ik(\pi + a)} - e^{-ik(\pi + a)} + 1 \right)$$

$$= \frac{1}{4\pi^2 k^2} \cdot \left(2 - 2\cos(k(\pi + a)) \right) = \frac{1}{2\pi^2 k^2} \cdot \left(1 - \cos(k(\pi + a)) \right)$$

$$\Rightarrow \sum_{k=-n}^{n} |c_k|^2 = \left(\frac{a + \pi}{2\pi} \right)^2 + \frac{1}{2\pi^2} \left(\sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{1}{k^2} - \sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{\cos(k(\pi + a))}{k^2} \right)$$

$$= \left(\frac{a + \pi}{2\pi} \right)^2 + \frac{1}{\pi^2} \left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{\cos(k(\pi + a))}{k^2} \right)$$

$n \to \infty$: (*) $= \frac{\pi^2}{6}$; (**) $\frac{a^2}{4} - \frac{\pi^2}{12}$

General formula: $x \in [0, 2\pi]$

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x - \pi)^2}{4} - \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \left(\frac{a + \pi}{2\pi} \right)^2 + \frac{1}{\pi^2} \left(\frac{\pi^2}{6} - \frac{a^2}{4} + \frac{\pi^2}{12} \right)$$

$$= \left(\frac{a + \pi}{2\pi} \right)^2 + \frac{1}{4} - \frac{a^2}{4\pi^2} = \frac{2a\pi + \pi^2}{4\pi^2} + \frac{1}{4}$$

$$= \frac{a}{2\pi} + \frac{1}{2} = \frac{1}{2\pi} \cdot (a + \pi) = \frac{1}{2\pi} \int_{-\pi}^{a} 1\, dx = \langle h_a, h_a \rangle$$

$$= \| h_a \|^2$$

Fourier Transform – Part 11

Let's prove: $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$, $x \in [0, 2\pi]$

Note:

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kx)) = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2}\cdot\left(e^{ikx} + e^{-ikx}\right) = \frac{1}{2}\sum_{k=-n}^{n} e^{ikx}$$

$$= \frac{1}{2} e^{-inx} \sum_{k=0}^{2n} \underbrace{e^{ikx}}_{q^k} \qquad q = e^{ix}$$

$$= \frac{1}{2} e^{-inx} \cdot \frac{1 - q^{2n+1}}{1 - q} \qquad \text{geometric sum formula } q \neq 1$$

$$= \frac{1}{2} \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}} \cdot \frac{-e^{-\frac{1}{2}ix}}{-e^{-\frac{1}{2}ix}}$$

$$= \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \cdot \frac{\frac{1}{2i}}{\frac{1}{2i}} = \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

for $x \in \mathbb{R} \setminus \{2\pi m \mid m \in \mathbb{Z}\}$

Lemma: $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2}$ for $x \in (0, 2\pi)$

and we have uniform convergence on interval $[\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$.

Proof:

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} = \sum_{k=1}^{n} \int_{\pi}^{x} \cos(kt))\,dt = \int_{\pi}^{x} \sum_{k=1}^{n} \cos(kt))\,dt$$

$$= \int_{\pi}^{x} \left(\frac{1}{2} \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} - \frac{1}{2}\right) dt$$

$$= \underbrace{\int_{\pi}^{x} \frac{\sin((n+\frac{1}{2})t)}{2\sin(\frac{1}{2}t)}\,dt}_{f_n(x)} - \frac{1}{2}(x - \pi)$$

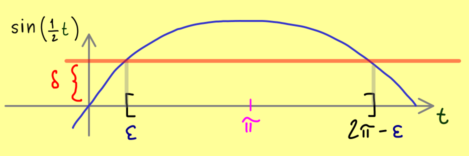
integration by parts: $\displaystyle f_n(x) = \int_{\pi}^{x} \underbrace{\frac{1}{2\sin(\frac{1}{2}t)}}_{u} \cdot \underbrace{\sin((n+\tfrac{1}{2})t)}_{v'}\,dt$ $\qquad u' = -\frac{1}{2} \frac{\frac{1}{2}\cos(\frac{1}{2}t)}{(\sin(\frac{1}{2}t))^2}$

$$v = \frac{1}{n+\frac{1}{2}} \cdot (-1) \cdot \cos((n+\tfrac{1}{2})t)$$

$$f_n(x) = \frac{1}{n+\frac{1}{2}} \cdot \frac{(-1)\cos((n+\frac{1}{2})t)}{2\sin(\frac{1}{2}t)} \Bigg|_{\pi}^{x} - \int_{\pi}^{x} \frac{1}{n+\frac{1}{2}} \frac{(-1)\cdot\cos((n+\frac{1}{2})t)\cos(\frac{1}{2}t)}{(-4)\cdot(\sin(\frac{1}{2}t))^2}\,dt$$

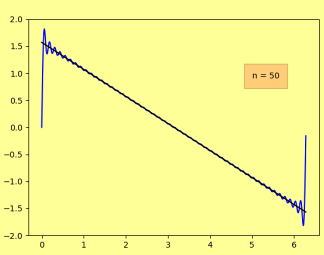
$$= \frac{1}{n+\frac{1}{2}} \Bigg(\underbrace{\frac{(-1)\cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}}_{= a(x)} - \underbrace{\frac{1}{4} \int_{\pi}^{x} \frac{\cos((n+\frac{1}{2})t)\cos(\frac{1}{2}t)}{(\sin(\frac{1}{2}t))^2}\,dt}_{b(x)} \Bigg)$$

For $\varepsilon > 0$, choose $x \in [\varepsilon, 2\pi - \varepsilon]$:



$$\| f_n \|_\infty \leq \frac{1}{n+\frac{1}{2}} \left(\|a\|_\infty + \|b\|_\infty \right)$$

$$\leq \frac{1}{n+\frac{1}{2}} \left(\frac{1}{2\delta} + \frac{1}{4\delta^2} \cdot \pi \right) \xrightarrow{n \to \infty} 0$$



Recall $\displaystyle f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} + \frac{1}{2}(x - \pi)$ □

Theorem: $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$, $x \in [0, 2\pi]$

← uniform convergence on $[0, 2\pi]$

Proof: For $\varepsilon > 0$, $x, x_0 \in [\varepsilon, 2\pi - \varepsilon]$: (use Lemma)

$$\int_{x_0}^{x} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k}\,dt = \int_{x_0}^{x} \frac{\pi - t}{2}\,dt = -\frac{(\pi - t)^2}{4}\Bigg|_{x_0}^{x} = -\frac{(x-\pi)^2}{4} + \underbrace{\frac{(x_0-\pi)^2}{4}}_{C_0}$$

uniform convergence → ‖

$$\sum_{k=1}^{\infty} \int_{x_0}^{x} \frac{\sin(kt)}{k}\,dt = \sum_{k=1}^{\infty} -\frac{\cos(kt))}{k^2}\Bigg|_{x_0}^{x} = -\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} + C_1$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C \qquad \leftarrow \text{calculate it!}$$

→ still uniform convergence on $[\varepsilon, 2\pi - \varepsilon]$

We know more:

(1) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}$ uniformly convergent on $[0, 2\pi]$

by Weierstrass M-test since $\displaystyle\left|\frac{\cos(kx))}{k^2}\right| \leq \frac{1}{k^2}$

$\displaystyle\Rightarrow [0, 2\pi] \ni x \mapsto \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}$ continuous function

(2) $\displaystyle [0, 2\pi] \ni x \mapsto \frac{(x-\pi)^2}{4} + C$ continuous function

(3) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$ for all $x \in (0, 2\pi)$

$\displaystyle\Rightarrow \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$ uniformly convergent on $[0, 2\pi]$

Find C:

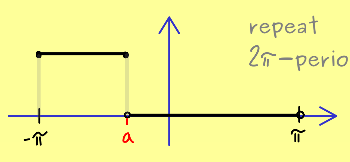
$$\int_{0}^{2\pi} \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}\,dx = \int_{0}^{2\pi} \left(\frac{(x-\pi)^2}{4} + C\right) dx = \underbrace{\frac{(x-\pi)^3}{12}\Bigg|_{0}^{2\pi}}_{\frac{\pi^3}{6}} + 2\pi \cdot C$$

‖ uniform convergence

$$\sum_{k=1}^{\infty} \int_{0}^{2\pi} \frac{\cos(kx))}{k^2}\,dx = 0 \qquad \Rightarrow \quad C = -\frac{\pi^2}{12}$$

Fourier Transform – Part 12

h_a :



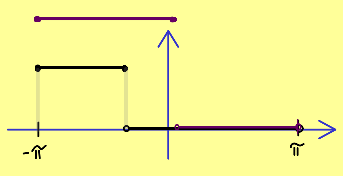
repeat 2π-periodically

$$h_a(x) = \begin{cases} 1, & x \in [-\pi, a] \\ 0, & x \in (a, \pi) \end{cases}$$

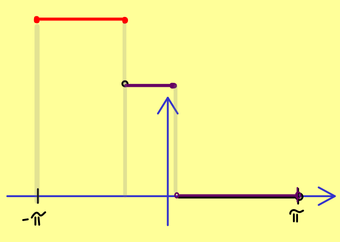
Parseval's identity holds for h_a for every possible a. (part 10)

Step functions: consider the complex vector space:

$$S_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \left\{ g \in \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \;\middle|\; \text{there are } m \in \mathbb{N},\ a_i \in [-\pi,\pi],\ \lambda_i \in \mathbb{C} \text{ such that: } g = \sum_{i=1}^{m} \lambda_i \cdot h_{a_i} \right\}$$



adding



Do we have Parseval's identity here?

Consider step function $g \in S_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

$$g = \sum_{i=1}^{m} \lambda_i \cdot h_{a_i}$$

$$c_k = \langle e_k, g \rangle = \left\langle e_k, \sum_{i=1}^{m} \lambda_i \cdot h_{a_i} \right\rangle = \sum_{i=1}^{m} \lambda_i \langle e_k, h_{a_i} \rangle$$

$$|c_k|^2 = \overline{c_k}\, c_k = \overline{\sum_{j=1}^{m} \lambda_j \langle e_k, h_{a_j} \rangle} \cdot \sum_{i=1}^{m} \lambda_i \langle e_k, h_{a_i} \rangle$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{m} \overline{\lambda_j}\, \lambda_i \langle h_{a_j}, e_k \rangle \langle e_k, h_{a_i} \rangle$$

$$\sum_{k=-n}^{n} |c_k|^2 = \sum_{i,j=1}^{m} \overline{\lambda_j}\, \lambda_i \left(\sum_{k=-n}^{n} \langle h_{a_j}, e_k \rangle \langle e_k, h_{a_i} \rangle \right) \xrightarrow{n \to \infty} \langle h_{a_j}, h_{a_i} \rangle$$

(part 9)

informal: $\left(\sum_{k=-\infty}^{\infty} |e_k\rangle\langle e_k| = \mathbb{1} \right)$

we have Parseval's identity for h_{a_j} and h_{a_i}

$$\Longrightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 = \sum_{i,j=1}^{m} \overline{\lambda_j}\, \lambda_i \langle h_{a_j}, h_{a_i} \rangle = \left\langle \sum_{j=1}^{m} \lambda_j \cdot h_{a_j}, \sum_{i=1}^{m} \lambda_i \cdot h_{a_i} \right\rangle$$

$$= \langle g, g \rangle = \|g\|^2$$

Result: Parseval's identity holds for $S_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \subseteq L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$.

Fourier Transform - Part 13

Theorem: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\cdot g(x)\, dx$

and ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k : x \mapsto e^{ikx}$.

For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ define: $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \underbrace{\langle e_k, f\rangle}_{c_k}$.

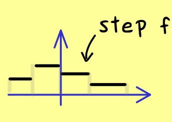
Then: $\| f - \mathcal{F}_n(f)\| \xrightarrow{n\to\infty} 0$ (L^2-norm)

(equivalent to Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f\rangle|^2$)

Fact: Continuous functions are dense in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$, which means:

For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ and $\varepsilon > 0$, there is a 2π-periodic continuous function $g : \mathbb{R} \to \mathbb{C}$ with $\|f - g\| < \varepsilon$. (L^2-norm)

Proposition: $\mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ is dense in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$.

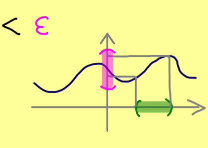


Proof: Let $\varepsilon > 0$, $f : [-\pi, \pi] \to \mathbb{C}$ square integrable.

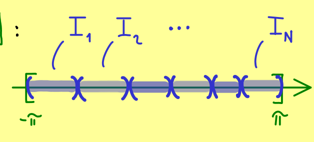
Then there is a continuous function $g : [-\pi,\pi] \to \mathbb{C}$ with $\|f - g\| < \varepsilon$.

$\overset{\text{domain compact}}{\Longrightarrow}$ g is uniformly continuous : for given $\varepsilon > 0$ there $\delta > 0$:

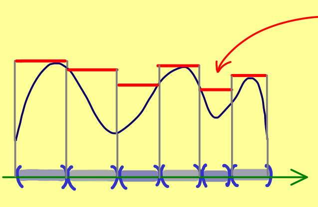
$$|x - y| < \delta \implies |g(x) - g(y)| < \varepsilon$$



Decompose $[-\pi, \pi]$:



$\text{length}(I_j) < \delta$



$c_j := \sup\{ g(x) \mid x \in \overline{I_j}\}$

define step function:

$h(x) = c_j$ for $x \in I_j$

We get: $|g(x) - h(x)| = |g(x) - g(y)|$ for $y \in \overline{I_j}$ ($x \in I_j$)

$< \varepsilon$ because $|x - y| < \delta$

In total: $\|f - h\| \leq \underbrace{\|f - g\|}_{<\varepsilon} + \|g - h\| < C \cdot \varepsilon$ (constant) $\square$

$\|g - h\| = \left(\int_{-\pi}^{\pi} \underbrace{|g(x) - h(x)|^2}_{<\varepsilon}\right)^{1/2}$

Theorem (see above): For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$: $\|f - \mathcal{F}_n(f)\| \xrightarrow{n\to\infty} 0$

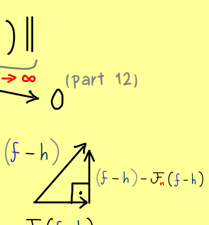
Proof: Let $\varepsilon > 0$, $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$. Choose $h \in \mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with $\|f - h\| < \varepsilon$.

Then: $\|f - \mathcal{F}_n(f)\| = \|f + h - h - \mathcal{F}_n(f) + \mathcal{F}_n(h) - \mathcal{F}_n(h)\|$

$$\leq \underbrace{\|(f-h) - \mathcal{F}_n(f-h)\|}_{\leq \|(f-h)\| < \varepsilon} + \underbrace{\|h - \mathcal{F}_n(h)\|}_{\xrightarrow{h\to\infty} 0 \text{ (part 12)}}$$

Pythagorean theorem:

$$\|(f-h) - \mathcal{F}_n(f-h)\|^2 + \|\mathcal{F}_n(f-h)\|^2 = \|(f-h)\|^2$$

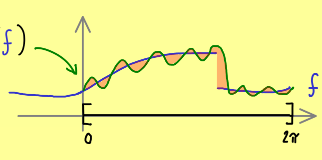


$$\implies \lim_{n\to\infty} \|f - \mathcal{F}_n(f)\| = 0$$

$\square$

The Bright Side of Mathematics

Fourier Transform - Part 14

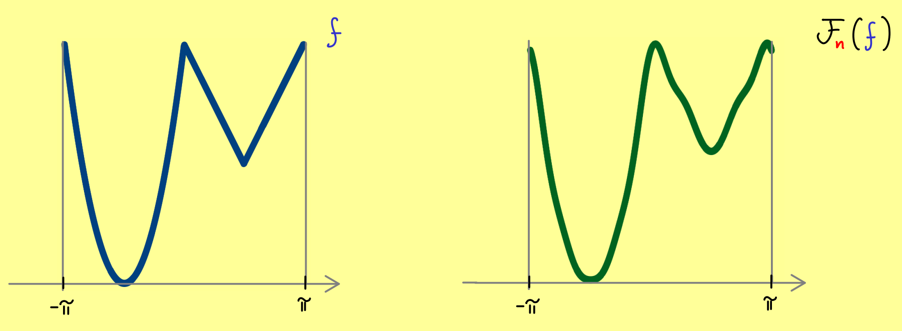


$\| f - \mathcal{F}_n(f) \|_{L^2} \xrightarrow{n \to \infty} 0$

not a pointwise convergence!

$\rightsquigarrow$ We can get uniform convergence for special functions

Example: continuous and piecewise C^1-function

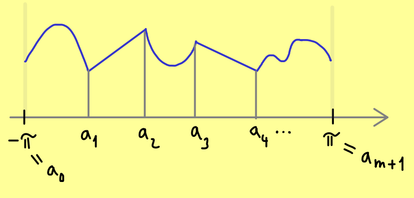


Supremum norm: $\| f \|_\infty := \sup_{x \in [-\pi,\pi]} |f(x)|$

$$\Rightarrow \int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} \| f \|_\infty^2 \, dx = 2\pi \cdot \| f \|_\infty^2$$

$$\Rightarrow \| f \|_{L^2} \leq \| f \|_\infty$$

Theorem: $f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic continuous function.



Assume there are finitely many points $(a_1, a_2, \dots, a_m)$ inside the interval $[-\pi, \pi]$ such that:

$$f|_{[a_j, a_{j+1}]} \in C^1 \quad \text{for all} \quad j \in \{0, 1, \dots, m\}$$

Then: $\| f - \mathcal{F}_n(f) \|_\infty \xrightarrow{n \to \infty} 0$

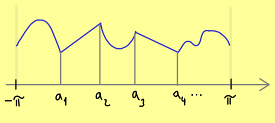
$$\left(\begin{array}{c} \mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \langle e_k, f \rangle \\ e_k: x \mapsto e^{ikx} \\ \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x) \, dx \end{array} \right)$$

The Bright Side of Mathematics

Fourier Transform - Part 15

Theorem: $f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic continuous function

and piecewise C^1-function:



there are finitely many points $(a_1, a_2, \dots, a_m)$

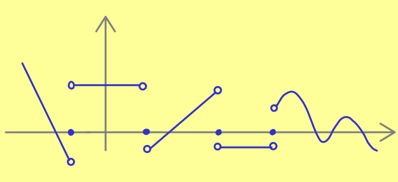
inside the interval $[-\pi, \pi]$ such that: $f|_{[a_j, a_{j+1}]} \in C^1$

for all $j \in \{0, 1, \dots, m\}$, $a_0 := -\pi$, $a_{m+1} := \pi$

Then: $\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ uniformly.

Proof: Consider the derivative function:

$$\tilde{f}(x) := \begin{cases} 0 , & x \in \{a_0, a_1, \dots, a_{m+1}\} \\ f'(x) , & \text{else} \end{cases}$$



→ piecewise continuous function $\in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

Parseval's identity: $\|\tilde{f}\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, \tilde{f} \rangle|^2 < \infty$

What about the Fourier coefficients of f? $(k \neq 0)$

$$c_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{e^{-ikx}}_{u'} \underbrace{f(x)}_{v} \, dx = \frac{1}{2\pi} \left(u \cdot v \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u v' \, dx \right)$$

$u = \frac{1}{-ik} e^{-ikx}$ (integration by parts)

$$= \frac{1}{2\pi} \left(0 + \frac{1}{ik} \int_{-\pi}^{\pi} e^{-ikx} \tilde{f}(x) \, dx \right) = \frac{1}{ik} \langle e_k, \tilde{f} \rangle$$

General inequality for real numbers: $x \cdot y \leq \frac{x^2 + y^2}{2}$

$$|c_k| = \frac{1}{k} |\langle e_k, \tilde{f} \rangle| \leq \frac{1}{2} \left(\frac{1}{k^2} + |\langle e_k, \tilde{f} \rangle|^2 \right)$$

$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |c_k| \leq \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k^2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |\langle e_k, \tilde{f} \rangle|^2 < \infty$$

$$\mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} \underbrace{e^{ikx} \cdot c_k}_{f_k(x)} \quad \text{with} \quad |f_k(x)| \leq M_k =: |c_k| , \quad \sum_{k=-\infty}^{\infty} M_k < \infty$$

$\overset{\text{Weierstrass M-Test}}{\Longrightarrow}$ $\sum_{k=-\infty}^{\infty} f_k$ uniformly convergent to a continuous function $h: [-\pi, \pi] \longrightarrow \mathbb{C}$

Status quo: $\|\mathcal{F}_n(f) - h\|_\infty \xrightarrow{n \to \infty} 0$, $\|\mathcal{F}_n(f) - f\|_{L^2} \xrightarrow{n \to \infty} 0$

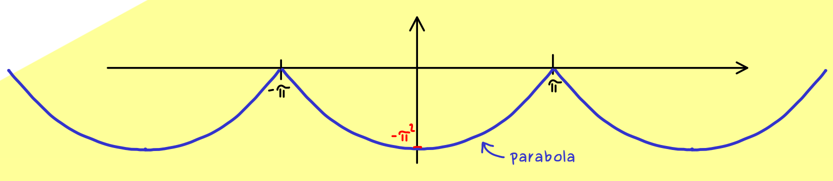
More estimates: $\|f - h\|_{L^2} \leq \|f - \mathcal{F}_n(f)\|_{L^2} + \underbrace{\|\mathcal{F}_n(f) - h\|_{L^2}}_{\leq \|\mathcal{F}_n(f) - h\|_\infty} \xrightarrow{n \to \infty} 0$

Hence: $\|f - h\|_{L^2} = 0 \overset{\text{continuous functions}}{\Longrightarrow} f = h$

Conclusion: $\|\mathcal{F}_n(f) - f\|_\infty \xrightarrow{n \to \infty} 0$ (uniform convergence of the Fourier series)

$\square$

Fourier Transform - Part 16



$\Rightarrow$ continuous + piecewise C^1-function

Example: $f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic with $f(x) = x^2 - \pi^2$ for $x \in [-\pi, \pi]$.

Let's calculate the Fourier coefficients: $c_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 - \pi^2)\, dx = \frac{1}{2\pi} \left(\frac{1}{3} x^3 - \pi^2 x \right) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \cdot 2 \cdot \left(\frac{1}{3}\pi^3 - \pi^3 \right) = -\frac{2}{3}\pi^2$$

For $k \neq 0$: $c_k = \frac{1}{ik} \langle e_k, f' \rangle$ (integration by parts, see part 15)

$$= \frac{1}{2\pi i k} \int_{-\pi}^{\pi} \underbrace{e^{-ikx}}_{u'} \underbrace{2 \cdot x}_{v} \, dx \quad \text{(integration by parts)}$$

$\rightsquigarrow u = \frac{1}{-ik} e^{-ikx}, \; v' = 2$

$$= \frac{1}{2\pi i k} \left(-\frac{1}{ik} e^{-ikx} \cdot 2 \cdot x \Big|_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} \left(-\frac{1}{ik} e^{-ikx} \right) \cdot 2\, dx}_{=0} \right)$$

$$= \frac{1}{\pi \cdot k^2} \left(\underbrace{e^{-ik\pi}}_{=(-1)^k} \pi - \underbrace{e^{ik\pi}}_{=(-1)^k} (-\pi) \right)$$

$$= \frac{2 \cdot (-1)^k}{k^2}$$

Fourier series: ($x \in [-\pi, \pi]$)

$$x^2 - \pi^2 = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = -\frac{2}{3}\pi^2 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{2 \cdot (-1)^k}{k^2} \underbrace{e^{ikx}}_{= \cos(kx) + i \cdot \sin(kx)}$$

$$= -\frac{2}{3}\pi^2 + 2 \cdot \sum_{k=1}^{\infty} \frac{2 \cdot (-1)^k}{k^2} \cos(kx)$$

For all $x \in [-\pi, \pi]$: $\quad x^2 - \frac{1}{3}\pi^2 = \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k \cos(kx)$ ← uniform convergence!

In particular for $x = 0$: $\quad -\frac{1}{3}\pi^2 = \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{1}{12}\pi^2}$$

Parseval's identity:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|_{L^2}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 - \pi^2)^2\, dx = \frac{8}{15}\pi^4$$

$$\| \quad |c_0|^2 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left| \frac{2 \cdot (-1)^k}{k^2} \right|^2$$

$$\| \quad \frac{4}{9} \cdot \pi^4 + 2 \cdot \sum_{k=1}^{\infty} \frac{4}{k^4} \qquad \Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}}$$

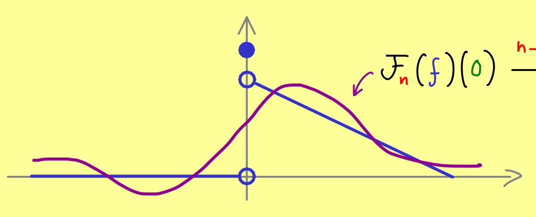
Fourier Transform - Part 17

$f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic

$f \in \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$ $\Longrightarrow$ $\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ (in L^2-norm)

$\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ (pointwisely)

continuous + piecewise C^1-function $\Longrightarrow$ $\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ (uniformly)



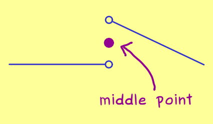
$\mathcal{F}_n(f)(0) \xrightarrow{n \to \infty}$?

Theorem: $f \in \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$, $\hat{x} \in [-\pi, \pi]$ with:

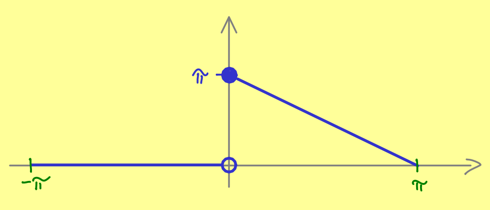
$f(\hat{x}^-) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} - \varepsilon)$ exists, $\lim_{\substack{h \to 0 \\ h < 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h}$ exists

$f(\hat{x}^+) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} + \varepsilon)$ exists, $\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h}$ exists

Then: $\mathcal{F}_n(f)(\hat{x}) \xrightarrow{n \to \infty} \frac{1}{2}\left(f(\hat{x}^+) + f(\hat{x}^-) \right)$



Example:



$$f(x) = \begin{cases} 0 , & x \in [-\pi, 0) \\ \pi - x , & x \in [0, \pi) \end{cases}$$

Fourier coefficients: $c_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_0^\pi e^{-ikx} (\pi - x) \, dx$

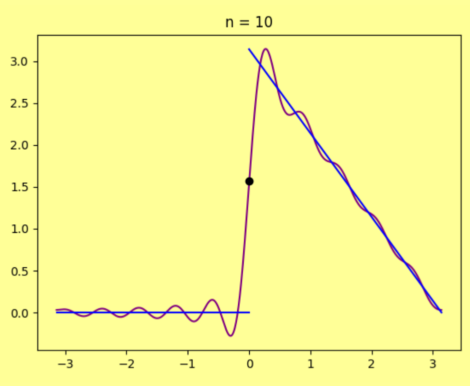
$$= \begin{cases} \frac{\pi}{4} , & k = 0 \\ \frac{1}{2\pi} \cdot \left(-\frac{1}{k^2}\left((-1)^k - 1\right) - i \frac{\pi}{k} \right) , & k \neq 0 \end{cases}$$

Fourier series: $\mathcal{F}_n(f)(x) = \frac{\pi}{4} + \sum_{\substack{k=-n \\ k \neq 0}}^{n} c_k \cdot e^{ikx}$

$a_k = c_k + c_{-k}$
$b_k = i \cdot (c_k - c_{-k})$

$$= \frac{\pi}{4} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$$

$a_k = \frac{1}{\pi} \cdot \frac{1}{k^2}\left(1 - (-1)^k\right)$, $b_k = \frac{1}{k}$



$\mathcal{F}_n(f)(0) \xrightarrow{n \to \infty} \frac{\pi}{2}$

$\Longrightarrow \frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{1}{\pi} \frac{1 - (-1)^k}{k^2}$

$\Longrightarrow \frac{\pi^2}{4} = \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k^2}$

The Bright Side of Mathematics

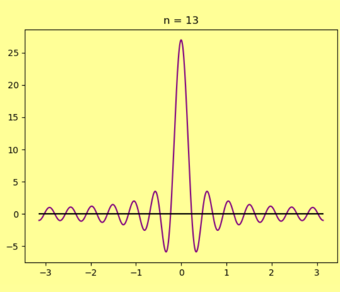
Fourier Transform - Part 18

Definition: The continuous function $\mathcal{D}_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, given by

$$\mathcal{D}_n(x) = \sum_{k=-n}^{n} e^{ikx} \underset{\text{part 11}}{=} 1 + 2 \cdot \sum_{k=1}^{n} \cos(kx) \underset{\text{part 11}}{=} \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

$2\pi \cdot m$ zeros for $m \in \mathbb{Z}$

is called the Dirichlet kernel.



2π-periodic

For Fourier series:

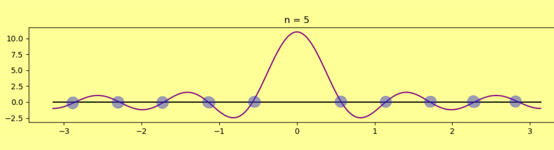
$$\mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} c_k \cdot e^{ikx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iky} f(y)\, dy \right) \cdot e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{ik(x-y)}\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \mathcal{D}_n(\underbrace{x-y}_{z})\, dy$$

$$= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-z)\, \mathcal{D}_n(z)\, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{D}_n(z)\, f(x-z)\, dz$$

$$= \left\langle \mathcal{D}_n, f(x-\cdot) \right\rangle = \frac{1}{2\pi} \left(\mathcal{D}_n * f \right)(x) \quad \text{(convolution)}$$

Properties: (1) $\mathcal{D}_n$ has exactly $2n$ zeros inside the interval $[-\pi, \pi]$

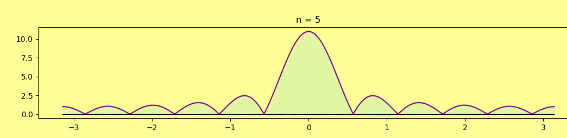


$$\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

(2) $$\int_{-\pi}^{\pi} \mathcal{D}_n(x)\, dx = \int_{-\pi}^{\pi} \left(1 + e^{ix} + e^{-ix} + e^{2ix} + e^{-2ix} + \dots + e^{nix} + e^{-nix}\right) dx$$

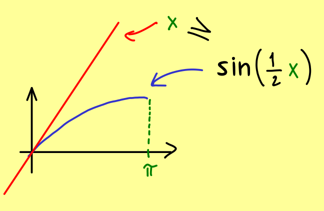
$$= 2\pi \qquad \Rightarrow \left\langle \mathcal{D}_n, 1 \right\rangle = 1$$

(3) $$\int_{-\pi}^{\pi} \left| \mathcal{D}_n(x) \right| dx \xrightarrow{n \to \infty} \infty$$



Proof of (3):

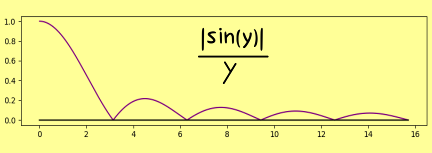
$$\left| \mathcal{D}_n(x) \right| = \frac{\left|\sin\left(\left(n+\frac{1}{2}\right)x\right)\right|}{\left|\sin\left(\frac{1}{2}x\right)\right|}$$



$$\geq \frac{\left|\sin\left(\left(n+\frac{1}{2}\right)x\right)\right|}{x} \quad \text{for all } x > 0$$

$$\int_{-\pi}^{\pi} \left| \mathcal{D}_n(x) \right| dx = 2 \cdot \int_{0}^{\pi} \left| \mathcal{D}_n(x) \right| dx \geq 2 \cdot \int_{0}^{\pi} \frac{\left|\sin\left(\overbrace{\left(n+\frac{1}{2}\right)x}^{y}\right)\right|}{x}\, dx$$

$$= 2 \cdot \int_{0}^{\left(n+\frac{1}{2}\right)\pi} \frac{|\sin(y)|}{y}\, dy \geq 2 \cdot \int_{0}^{n \cdot \pi} \frac{|\sin(y)|}{y}\, dy$$



$$= 2 \cdot \sum_{k=1}^{n} \int_{(k-1)\cdot\pi}^{k \cdot \pi} \frac{|\sin(y)|}{y}\, dy$$

maximal $k \cdot \pi$

$$\geq 2 \cdot \sum_{k=1}^{n} \int_{(k-1)\cdot\pi}^{k \cdot \pi} \frac{|\sin(y)|}{k \cdot \pi}\, dy$$

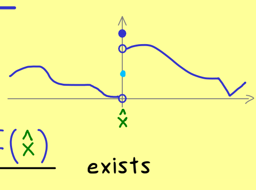
$$= 2 \cdot \sum_{k=1}^{n} \frac{1}{k\pi} \underbrace{\int_{(k-1)\cdot\pi}^{k \cdot \pi} |\sin(y)|\, dy}_{=1} = \text{const} \cdot \sum_{k=1}^{n} \frac{1}{k}$$

$$\xrightarrow{n \to \infty} \infty$$

The Bright Side of Mathematics

Fourier Transform - Part 19

Theorem: $f \in \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$, $\hat{x} \in [-\pi, \pi]$ with:



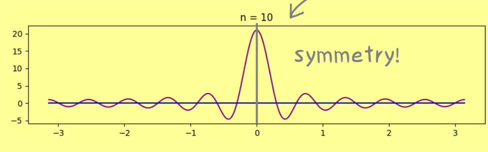
$$f(\hat{x}^-) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} - \varepsilon) \text{ exists}, \quad \lim_{\substack{h \to 0 \\ h < 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} \text{ exists}$$

$$f(\hat{x}^+) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} + \varepsilon) \text{ exists}, \quad \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} \text{ exists}$$

Then: $$\mathcal{F}_n(f)(\hat{x}) \xrightarrow{n \to \infty} \frac{1}{2}\Big(f(\hat{x}^+) + f(\hat{x}^-) \Big) =: M$$

Proof: Dirichlet kernel: $D_n(x) = \dfrac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{1}{2}x\right)}$ gives $\mathcal{F}_n(f)(\hat{x}) = \langle D_n, f(\hat{x} - \cdot) \rangle$

and $\langle D_n, M \rangle = M$



Use symmetry:

$$\langle D_n, f(\hat{x} - \cdot) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) f(\hat{x} - x)\, dx$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{0} D_n(x) f(\hat{x} \underbrace{- x}_{y})\, dx + \int_{0}^{\pi} D_n(x) f(\hat{x} - x)\, dx \right)$$

$$= \frac{1}{2\pi} \left(\int_{0}^{\pi} D_n(y) f(\hat{x} + y)\, dy + \int_{0}^{\pi} D_n(x) f(\hat{x} - x)\, dx \right)$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} D_n(y) \Big(f(\hat{x}+y) + f(\hat{x}-y) \Big)\, dy$$

Pointwise limit: $\mathcal{F}_n(f)(\hat{x}) - M = \langle D_n, f(\hat{x} - \cdot) \rangle - \langle D_n, M \rangle$

$$= \frac{1}{2\pi} \int_0^{\pi} D_n(y) \Big(f(\hat{x}+y) + f(\hat{x}-y) \Big)\, dy - \frac{1}{2\pi} \int_0^{\pi} D_n(y) \underbrace{2 \cdot M}_{= f(\hat{x}^+) + f(\hat{x}^-)}\, dy$$

$$= \frac{1}{2\pi} \int_0^{\pi} D_n(y) \Big(f(\hat{x}+y) - f(\hat{x}^+) + f(\hat{x}-y) - f(\hat{x}^-) \Big)\, dy$$

$$= \frac{1}{2\pi} \int_0^{\pi} \sin\left((n+\tfrac{1}{2})y\right) \underbrace{\frac{f(\hat{x}+y) - f(\hat{x}^+) + f(\hat{x}-y) - f(\hat{x}^-)}{\sin(\frac{1}{2}y)}}_{g(y)}\, dy$$

In the case that $g \in L^2_{2\pi\text{-per}}$, we get:

$$\frac{1}{2\pi} \int_0^{\pi} \underbrace{\sin\left((n+\tfrac{1}{2})y\right)}_{= \frac{1}{2i}\cdot\left(e^{iny} e^{i\frac{1}{2}y} - e^{-iny} e^{-i\frac{1}{2}y}\right)} g(y)\, dy$$

$$= \langle e_{-n}, g_1 \rangle + \langle e_n, g_2 \rangle \xrightarrow[\text{part 8}]{n \to \infty} 0$$

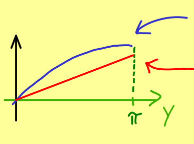
(Bessel's inequality)

g_1, g_2: $L^2_{2\pi\text{-per}}$-functions

Show that $g \in L^2_{2\pi\text{-per}}$:

$$g(y) = \begin{cases} \dfrac{f(\hat{x}+y) - f(\hat{x}^+)}{\sin(\frac{1}{2}y)} + \dfrac{f(\hat{x}-y) - f(\hat{x}^-)}{\sin(\frac{1}{2}y)}, & y \in (0, \pi) \\ 0, & y \in [-\pi, 0] \end{cases}$$

Does $g(y)$ explode for $y \to 0^+$?



$$\sin(\tfrac{1}{2}y) \geq \tfrac{1}{4}y \implies \left| \frac{f(\hat{x}+y) - f(\hat{x}^+)}{\sin(\frac{1}{2}y)} \right| \leq 4 \cdot \left| \frac{f(\hat{x}+y) - f(\hat{x}^+)}{y} \right| \xrightarrow{y \to 0^+} 4 \cdot |C^+|$$

because $$\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} =: C^+$$

and $$\left| \frac{f(\hat{x}-y) - f(\hat{x}^-)}{\sin(\frac{1}{2}y)} \right| \leq 4 \cdot \left| \frac{f(\hat{x}-y) - f(\hat{x}^-)}{y} \right| \xrightarrow{y \to 0^+} 4 \cdot |C^-|$$

because $$\lim_{\substack{h \to 0 \\ h < 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} =: C^-$$ □