Fourier Transform – Part 5

$$\mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) = \{ f : \mathbb{R} \to \mathbb{C} \mid f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R} \}$$

$$\mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \operatorname{Span}\Big(x \mapsto \tfrac{1}{\sqrt{2}},\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ x \mapsto \cos(3x), \ldots,$$
$$x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x), \ldots \Big)$$

↳ inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx$

Let's take integrable functions:

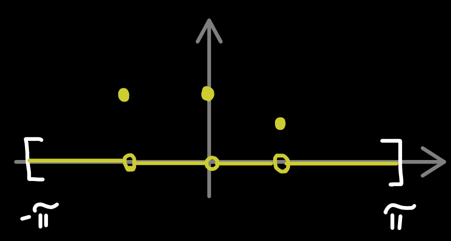
$$\mathcal{L}^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) = \Big\{ f \in \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \ \Big|\ \underbrace{\int_{-\pi}^{\pi} |f(x)|\, dx < \infty}_{f \text{ integrable with respect to Lebesgue measure on } [-\pi,\pi]} \Big\}$$

↳ complex vector space

norm? $\|f\|_1 := \int_{-\pi}^{\pi} |f(x)|\, dx$

↳ not a norm on $\mathcal{L}^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

problem:



solution: equivalence relation $f \sim g :\Longleftrightarrow \|f-g\|_1 = 0$

set of all equivalence classes: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \mathcal{L}^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) / \sim$

↳ complex vector space

$\|[f]\|_1 := \|f\|_1$

↳ norm!

identify: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

Let's take square-integrable functions:

$$\mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) = \left\{ f \in \mathcal{F}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \;\middle|\; \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty \right\}$$

norm? $\quad \|f\|_2 := \sqrt{\int_{-\pi}^{\pi} |f(x)|^2 \, dx}$

solution: equivalence relation $\quad f \sim g \;:\Longleftrightarrow\; \|f-g\|_2 = 0$

set of all equivalence classes: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) := \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) / \sim$

complex vector space with inner product