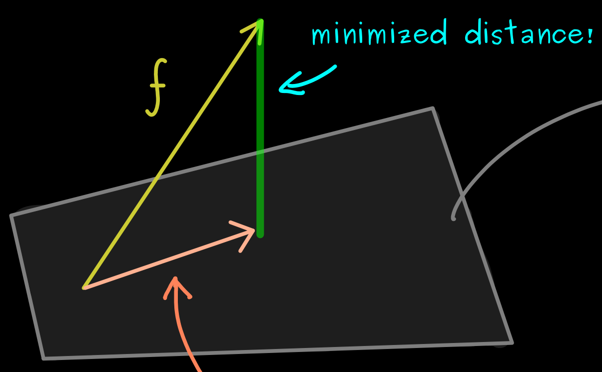
Fourier Transform – Part 6

We know: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

inner product: $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \overline{f(x)}\cdot g(x)\,dx$

Orthogonality: $\mathcal{B}_n = \big(x \mapsto \frac{1}{\sqrt{2}},\ x\mapsto \cos(x),\ x\mapsto\cos(2x),\ \dots,\ x\mapsto \cos(nx),$
$x\mapsto \sin(x),\ x\mapsto\sin(2x),\ x\mapsto\sin(3x),\ \dots,\ x\mapsto\sin(nx)\big)$

ONS in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ for every $n\in\mathbb{N}$



U_n finite-dimensional subspace spanned by $\mathcal{B}_n$

write: $\mathcal{B}_n = (h_1, h_2, \dots, h_N)$, $N = 2n+1$

orthogonal projection of f onto U_n:

$$\mathcal{F}_n(f) = \sum_{k=1}^{N} h_k \underbrace{\langle h_k, f\rangle}_{\text{Fourier coefficients}}$$

Definition: $\mathcal{F}_n(f)(x) = \tilde{a}_0\frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k\cdot\cos(k\cdot x) + \sum_{k=1}^{n} b_k\cdot\sin(k\cdot x)$

with $\tilde{a}_0 = \langle x\mapsto \frac{1}{\sqrt{2}}, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}f(x)\,dx$

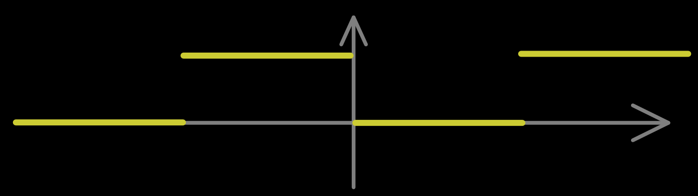
$a_k = \langle x\mapsto \cos(k\cdot x), f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(k\cdot x)\,f(x)\,dx$

$b_k = \langle x\mapsto \sin(k\cdot x), f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(k\cdot x)\,f(x)\,dx$

The map $n \mapsto \mathcal{F}_n(f)(x)$ (with $x\in\mathbb{R}$)

is called the Fourier series of $f\in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ (can be extended to $f\in L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$)

<u>Example</u>: $f : \mathbb{R} \to \mathbb{C}$, $\quad f(x) = \begin{cases} 1 , & x \in (-\pi, 0) \\ 0 , & x \in [0, \pi] \end{cases}$



$$\tilde{a}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(k \cdot x)\, f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \cos(k \cdot x)\, dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(k \cdot x)\, f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \sin(k \cdot x)\, dx = \frac{1}{\pi} \left(-\frac{1}{k} \cos(k \cdot x) \right) \Big|_{-\pi}^{0}$$

$$= \begin{cases} 0 , & k \text{ even} \\ -\frac{2}{\pi k} , & k \text{ odd} \end{cases}$$

Fourier series: $\quad \frac{1}{2} + \frac{-2}{\pi} \sin(x) + \frac{-2}{\pi 3} \cdot \sin(3 \cdot x) + \frac{-2}{\pi 5} \cdot \sin(5 \cdot x) + \cdots$