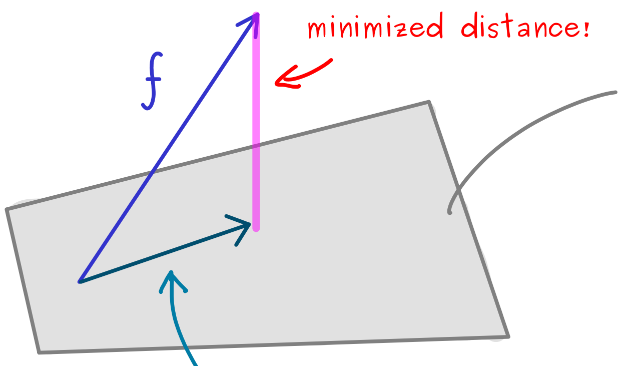
Fourier Transform – Part 6

We know: $L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

inner product: $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \overline{f(x)}\cdot g(x)\, dx$

Orthogonality: $\mathcal{B}_n = \Big(x \mapsto \frac{1}{\sqrt{2}},\ x \mapsto \cos(x),\ x \mapsto \cos(2x),\ \dots,\ x \mapsto \cos(nx),$
$x \mapsto \sin(x),\ x \mapsto \sin(2x),\ x \mapsto \sin(3x),\ \dots,\ x \mapsto \sin(nx) \Big)$

ONS in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ for every $n \in \mathbb{N}$



U_n finite-dimensional subspace spanned by $\mathcal{B}_n$

write: $\mathcal{B}_n = (h_1, h_2, \dots, h_N)$, $N = 2n+1$

orthogonal projection of f onto U_n:

$$\mathcal{F}_n(f) = \sum_{k=1}^{N} h_k \underbrace{\langle h_k, f\rangle}_{\text{Fourier coefficients}}$$

Definition: $\mathcal{F}_n(f)(x) = \tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cdot \cos(k\cdot x) + \sum_{k=1}^{n} b_k \cdot \sin(k\cdot x)$

with $\tilde{a}_0 = \langle x \mapsto \frac{1}{\sqrt{2}}, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\, dx$

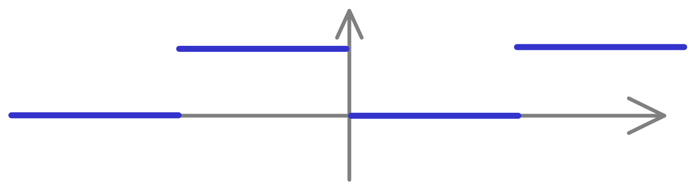
$a_k = \langle x \mapsto \cos(k\cdot x), f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k\cdot x) f(x)\, dx$

$b_k = \langle x \mapsto \sin(k\cdot x), f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(k\cdot x) f(x)\, dx$

The map $n \mapsto \mathcal{F}_n(f)(x)$ (with $x \in \mathbb{R}$)

is called the Fourier series of $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ (can be extended to $f \in L^1_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$)

Example: $f: \mathbb{R} \to \mathbb{C}$, $\quad f(x) = \begin{cases} 1, & x \in (-\pi, 0) \\ 0, & x \in [0, \pi] \end{cases}$



$$\tilde{a}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \frac{1}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(k \cdot x)\, f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \cos(k \cdot x)\, dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(k \cdot x)\, f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \sin(k \cdot x)\, dx = \frac{1}{\pi} \left(-\frac{1}{k} \cos(k \cdot x) \right) \Big|_{-\pi}^{0}$$

$$= \begin{cases} 0, & k \text{ even} \\ -\frac{2}{\pi k}, & k \text{ odd} \end{cases}$$

Fourier series: $\quad \frac{1}{2} + \frac{-2}{\pi} \sin(x) + \frac{-2}{\pi 3} \cdot \sin(3 \cdot x) + \frac{-2}{\pi 5} \cdot \sin(5 \cdot x) + \dots$