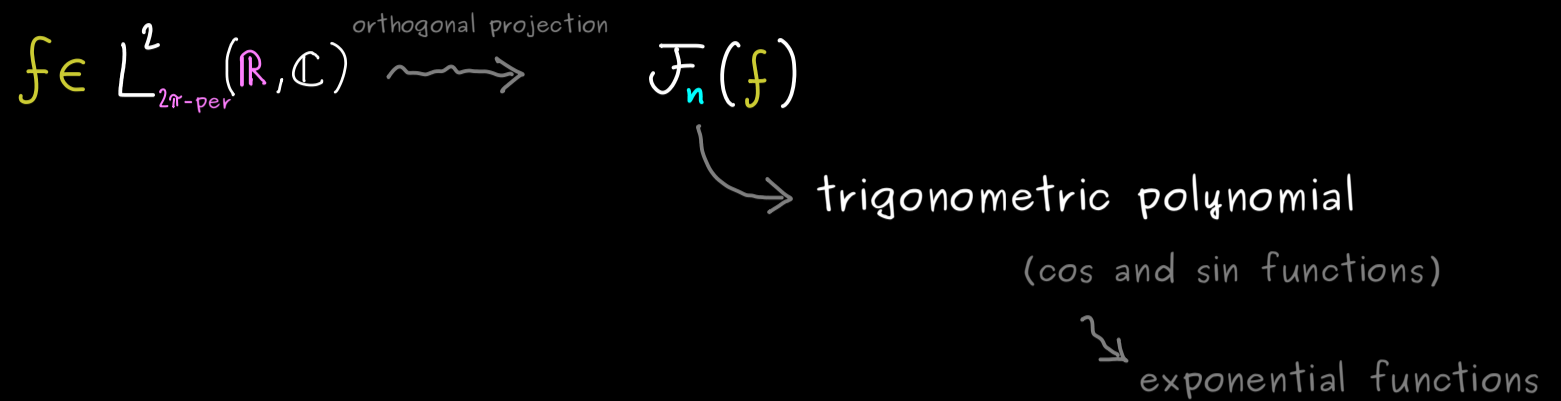




Fourier Transform - Part 7



Euler's formula: $\cos(x) = \frac{1}{2} (e^{ix} + e^{-ix})$

$$\sin(x) = \frac{1}{2i} (e^{ix} - e^{-ix})$$

Example:

$$A \cdot \cos(x) + B \cdot \cos(2x) + C \sin(2x), \quad A, B, C \in \mathbb{C}$$

$$= \frac{A}{2} (e^{ix} + e^{-ix}) + \frac{B}{2} (e^{i2x} + e^{-i2x}) + \frac{C}{2i} (e^{i2x} - e^{-i2x})$$

$$= \frac{A}{2} \cdot e^{ix} + \frac{A}{2} \cdot e^{-ix} + \left(\frac{B}{2} + \frac{C}{2i}\right) e^{i2x} + \left(\frac{B}{2} - \frac{C}{2i}\right) e^{-i2x}$$

complex linear combination!

Remember: In $\mathcal{P}_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$:

$$\text{Span} \left(x \mapsto \frac{1}{\sqrt{2}}, x \mapsto \cos(x), x \mapsto \cos(2x), \dots, x \mapsto \cos(nx), \right. \\ \left. x \mapsto \sin(x), x \mapsto \sin(2x), x \mapsto \sin(3x), \dots, x \mapsto \sin(nx) \right)$$

$$= \text{Span} \left(x \mapsto e^{-inx}, \dots, x \mapsto e^{-ix}, x \mapsto e^{i0x}, x \mapsto e^{ix}, \dots, x \mapsto e^{inx} \right)$$

and $\tilde{a}_0 \frac{1}{\sqrt{2}} + \sum_{k=1}^n a_k \cdot \cos(k \cdot x) + \sum_{k=1}^n b_k \cdot \sin(k \cdot x) = \sum_{k=-n}^n c_k e^{ikx}$

$$\text{with } c_k = \begin{cases} \frac{1}{2} \left(a_k + \frac{b_k}{i} \right), & \text{for } k > 0 \\ \tilde{a}_0 \frac{1}{\sqrt{2}} & \text{for } k = 0 \\ \frac{1}{2} \left(a_{-k} - \frac{b_{-k}}{i} \right), & \text{for } k < 0 \end{cases}$$

Result: Take $L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

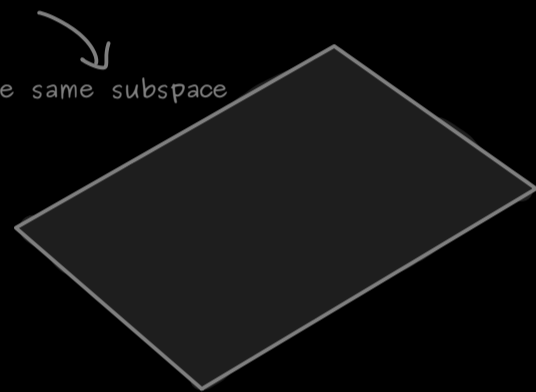
with inner product: $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x) dx$

best factor for exponential functions

ONS: $\mathcal{B}_n = \left(x \mapsto 1, x \mapsto \sqrt{2} \cos(x), x \mapsto \sqrt{2} \cos(2x), x \mapsto \sqrt{2} \cos(3x), \dots, x \mapsto \sqrt{2} \cos(nx), \right. \\ \left. x \mapsto \sqrt{2} \sin(x), x \mapsto \sqrt{2} \sin(2x), x \mapsto \sqrt{2} \sin(3x), \dots, x \mapsto \sqrt{2} \sin(nx) \right)$

ONS: $\mathcal{E}_n = \left(x \mapsto e^{ikx} \right)_{k=-n, \dots, n} = \left(e_k \right)_{k=-n, \dots, n}$

they span the same subspace



For $f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$: $\mathcal{F}_n(f) = \sum_{k=-n}^n e_k \underbrace{\langle e_k, f \rangle}_{\text{Fourier coefficients}}$

Fourier coefficients

$\Rightarrow \mathcal{F}_n(f)(x) = \sum_{k=-n}^n c_k e^{ikx}, \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$

The map $f \mapsto \mathcal{F}_n(f)$ is called the Fourier series of $f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

(with complex coefficients)