Fourier Transform – Part 7

$$f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \overset{\text{orthogonal projection}}{\rightsquigarrow} \mathcal{F}_n(f)$$

trigonometric polynomial

(cos and sin functions)

exponential functions

Euler's formula:

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Example:

$$A\cdot\cos(x) + B\cdot\cos(2x) + C\,\sin(2x)\,, \qquad A,B,C \in \mathbb{C}$$

$$= \frac{A}{2}\left(e^{ix} + e^{-ix}\right) + \frac{B}{2}\left(e^{i2x} + e^{-i2x}\right) + \frac{C}{2i}\left(e^{i2x} - e^{-i2x}\right)$$

$$= \frac{A}{2}\cdot e^{ix} + \frac{A}{2}\cdot e^{-ix} + \left(\frac{B}{2} + \frac{C}{2i}\right)e^{i2x} + \left(\frac{B}{2} - \frac{C}{2i}\right)e^{-i2x}$$

complex linear combination!

Remember: In $\mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$:

$$\mathrm{Span}\Big(x \mapsto \tfrac{1}{\sqrt{2}},\ x\mapsto\cos(x),\ x\mapsto\cos(2x),\ \dots,\ x\mapsto\cos(nx),$$
$$x\mapsto\sin(x),\ x\mapsto\sin(2x),\ x\mapsto\sin(3x),\ \dots,\ x\mapsto\sin(nx)\Big)$$

$$= \mathrm{Span}\left(x\mapsto e^{-inx},\dots,x\mapsto e^{-ix},\ x\mapsto e^{i0\cdot x},\ x\mapsto e^{ix},\dots,x\mapsto e^{inx}\right)$$

and

$$\tilde{a}_0\frac{1}{\sqrt{2}} + \sum_{k=1}^{n} a_k\cdot\cos(k\cdot x) + \sum_{k=1}^{n} b_k\cdot\sin(k\cdot x) = \sum_{k=-n}^{n} c_k\, e^{ikx}$$

with
$$c_k = \begin{cases} \frac{1}{2}\left(a_k + \frac{b_k}{i}\right), & \text{for } k>0 \\ \tilde{a}_0\frac{1}{\sqrt{2}} & \text{for } k=0 \\ \frac{1}{2}\left(a_{-k} - \frac{b_{-k}}{i}\right), & \text{for } k<0 \end{cases}$$

Result: Take $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C}) \supseteq \mathcal{P}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

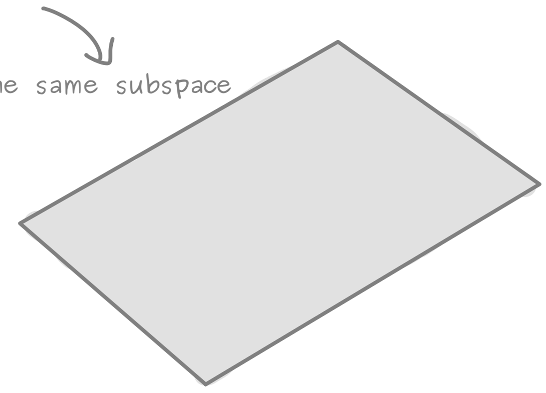
with inner product: $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x)\, dx$

best factor for exponential functions

ONS: $\mathcal{B}_n = \Big(x \mapsto 1,\ x \mapsto \sqrt{2}\cos(x),\ x \mapsto \sqrt{2}\cos(2x),\ x \mapsto \sqrt{2}\cos(3x), \dots, x \mapsto \sqrt{2}\cos(nx),$
$x \mapsto \sqrt{2}\sin(x),\ x \mapsto \sqrt{2}\sin(2x),\ x \mapsto \sqrt{2}\sin(3x), \dots, x \mapsto \sqrt{2}\sin(nx) \Big)$

ONS: $\mathcal{E}_n = \left(x \mapsto e^{ikx} \right)_{k=-n,\dots,n} = \left(e_k \right)_{k=-n,\dots,n}$

they span the same subspace



For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$: $\quad \mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \underbrace{\langle e_k, f \rangle}_{\text{Fourier coefficients}}$

$$\Rightarrow \quad \mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} c_k e^{ikx}, \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$$

The map $n \mapsto \mathcal{F}_n(f)$ is called the Fourier series of $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$

(with complex coefficients)