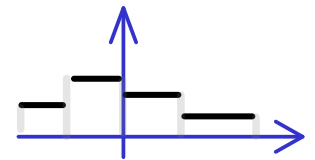


## Fourier Transform - Part 10

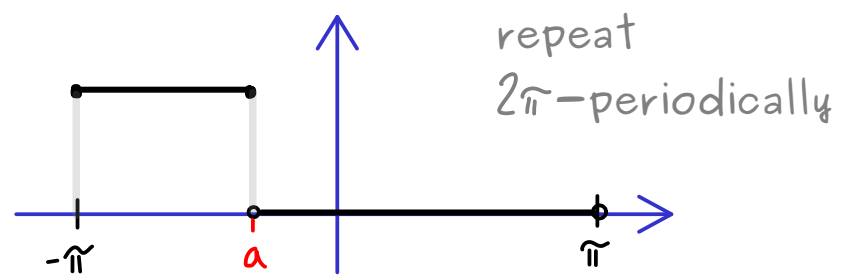
For proving Parseval's identity  $\rightsquigarrow$  step functions



Most important step function:

$$h_a(x) = \begin{cases} 1, & x \in [-\pi, a] \\ 0, & x \in (a, \pi) \end{cases}$$

for every  $a \in [-\pi, \pi]$



Fourier series for this example:

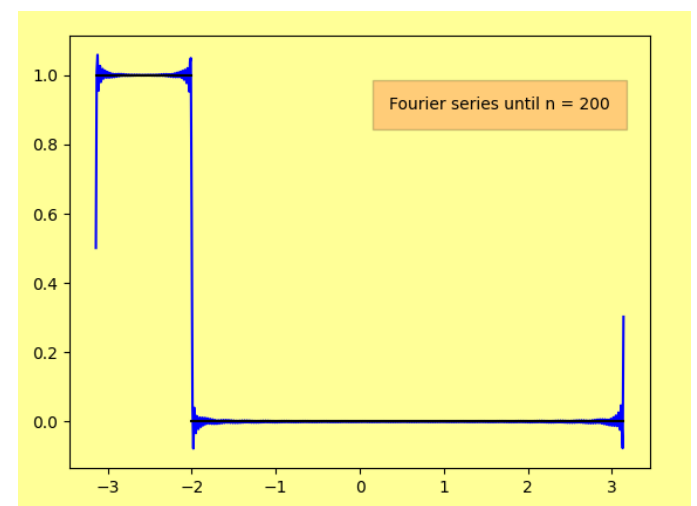
$$c_k = \langle e_k, h_a \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} h_a(x) dx = \frac{1}{2\pi} \int_{-\pi}^a e^{-ikx} dx$$
$$= \begin{cases} \frac{a + \pi}{2\pi}, & k = 0 \\ \frac{1}{2\pi(-ik)} (e^{-ika} - e^{ik\pi}), & k \neq 0 \end{cases}$$



Visualization:

$$a_k = 2 \cdot \text{Re}(c_k)$$

$$b_k = -2 \cdot \text{Im}(c_k)$$



Show Parseval's identity:

$$\begin{aligned}
 k \neq 0: \quad |c_k|^2 &= \frac{1}{2\tilde{\pi}(-ik)} \left( e^{-ika} - e^{ik\tilde{\pi}} \right) \overline{\frac{1}{2\tilde{\pi}(-ik)} \left( e^{-ika} - e^{ik\tilde{\pi}} \right)} \\
 &= \frac{1}{4\tilde{\pi}^2 k^2} \cdot \left( e^{-ika} - e^{ik\tilde{\pi}} \right) \cdot \left( e^{ika} - e^{-ik\tilde{\pi}} \right) \\
 &= \frac{1}{4\tilde{\pi}^2 k^2} \cdot \left( 1 - e^{ik(\tilde{\pi}+a)} - e^{-ik(\tilde{\pi}+a)} + 1 \right) \\
 &= \frac{1}{4\tilde{\pi}^2 k^2} \cdot \left( 2 - 2 \cos(k(\tilde{\pi}+a)) \right) = \frac{1}{2\tilde{\pi}^2 k^2} \cdot \left( 1 - \cos(k(\tilde{\pi}+a)) \right)
 \end{aligned}$$

$$\Rightarrow \sum_{k=-n}^n |c_k|^2 = \left( \frac{a + \tilde{\pi}}{2\tilde{\pi}} \right)^2 + \frac{1}{2\tilde{\pi}^2} \left( \sum_{\substack{k=-n \\ k \neq 0}}^n \frac{1}{k^2} - \sum_{\substack{k=-n \\ k \neq 0}}^n \frac{\cos(k(\tilde{\pi}+a))}{k^2} \right)$$

$$= \left( \frac{a + \tilde{\pi}}{2\tilde{\pi}} \right)^2 + \frac{1}{\tilde{\pi}^2} \left( \sum_{k=1}^n \frac{1}{k^2} - \sum_{k=1}^n \frac{\cos(k(\tilde{\pi}+a))}{k^2} \right)$$

General formula:  $x \in [0, 2\tilde{\pi}]$

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{(x - \tilde{\pi})^2}{4} - \frac{\tilde{\pi}^2}{12}$$

$$\begin{array}{ccc}
 \downarrow n \rightarrow \infty & & \downarrow n \rightarrow \infty \\
 (*) & & (**) \\
 = \frac{\tilde{\pi}^2}{6} & & \frac{a^2}{4} - \frac{\tilde{\pi}^2}{12}
 \end{array}$$

$$\begin{aligned}
 \Rightarrow \sum_{k=-\infty}^{\infty} |c_k|^2 &= \left( \frac{a + \tilde{\pi}}{2\tilde{\pi}} \right)^2 + \frac{1}{\tilde{\pi}^2} \left( \frac{\tilde{\pi}^2}{6} - \frac{a^2}{4} + \frac{\tilde{\pi}^2}{12} \right) \\
 &= \left( \frac{a + \tilde{\pi}}{2\tilde{\pi}} \right)^2 + \frac{1}{4} - \frac{a^2}{4\tilde{\pi}^2} = \frac{2a\tilde{\pi} + \tilde{\pi}^2}{4\tilde{\pi}^2} + \frac{1}{4} \\
 &= \frac{a}{2\tilde{\pi}} + \frac{1}{2} = \frac{1}{2\tilde{\pi}} \cdot (a + \tilde{\pi}) = \frac{1}{2\tilde{\pi}} \int_{-\tilde{\pi}}^a 1 \, dx = \langle h_a, h_a \rangle \\
 &= \|h_a\|^2
 \end{aligned}$$