The Bright Side of Mathematics

Fourier Transform – Part 11

Let's prove: $\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$, $x \in [0, 2\pi]$

Note:

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kx)) = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2} \cdot \left(e^{ikx} + e^{-ikx}\right) = \frac{1}{2} \sum_{k=-n}^{n} e^{ikx}$$

$$= \frac{1}{2} e^{-inx} \sum_{k=0}^{2n} \underbrace{e^{ikx}}_{q^k} \quad \rightarrow \quad q = e^{ix}$$

$$= \frac{1}{2} e^{-inx} \cdot \frac{1 - q^{2n+1}}{1-q} \quad \leftarrow \text{geometric sum formula} \quad q \neq 1$$

$$= \frac{1}{2} \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}} \cdot \frac{-e^{-\frac{1}{2}ix}}{-e^{-\frac{1}{2}ix}}$$

$$= \frac{1}{2} \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \cdot \frac{\frac{1}{2i}}{\frac{1}{2i}} = \frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

for $x \in \mathbb{R} \setminus \{2\pi m \mid m \in \mathbb{Z}\}$

Lemma: $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2}$ for $x \in (0, 2\pi)$

and we have uniform convergence on interval $[\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$.

Proof:

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} = \sum_{k=1}^{n} \int_{\pi}^{x} \cos(kt))\,dt = \int_{\pi}^{x} \sum_{k=1}^{n} \cos(kt))\,dt$$

$$= \int_{\pi}^{x} \left(\frac{1}{2} \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} - \frac{1}{2} \right) dt$$

$$= \underbrace{\int_{\pi}^{x} \frac{\sin((n+\frac{1}{2})t)}{2\sin(\frac{1}{2}t)}\,dt}_{f_n(x)} - \frac{1}{2}(x - \pi)$$

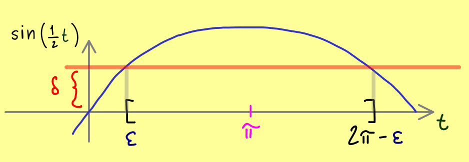
integration by parts: $f_n(x) = \int_{\pi}^{x} \underbrace{\frac{1}{2\sin(\frac{1}{2}t)}}_{u} \cdot \underbrace{\sin((n+\frac{1}{2})t)}_{v'}\,dt$ $\quad u' = -\frac{1}{2} \frac{\frac{1}{2}\cos(\frac{1}{2}t)}{\left(\sin(\frac{1}{2}t)\right)^2}$

$$v = \frac{1}{n+\frac{1}{2}} \cdot (-1) \cdot \cos((n+\tfrac{1}{2})t)$$

$$f_n(x) = \frac{1}{n+\frac{1}{2}} \cdot \frac{(-1)\cos((n+\frac{1}{2})t)}{2\sin(\frac{1}{2}t)} \Bigg|_{\pi}^{x} - \int_{\pi}^{x} \frac{1}{n+\frac{1}{2}} \frac{(-1)\cdot\cos((n+\frac{1}{2})t)\cos(\frac{1}{2}t)}{(-4)\cdot\left(\sin(\frac{1}{2}t)\right)^2}\,dt$$

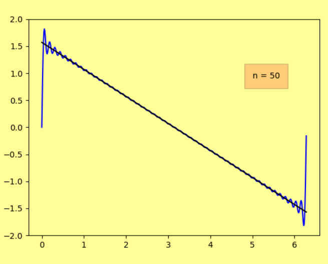
$$= \frac{1}{n+\frac{1}{2}} \left(\underbrace{\frac{(-1)\cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}}_{=a(x)} - \underbrace{\frac{1}{4} \int_{\pi}^{x} \frac{\cos((n+\frac{1}{2})t)\cos(\frac{1}{2}t)}{\left(\sin(\frac{1}{2}t)\right)^2}\,dt}_{b(x)} \right)$$

For $\varepsilon > 0$, choose $x \in [\varepsilon, 2\pi - \varepsilon]$:



$$\|f_n\|_\infty \leq \frac{1}{n+\frac{1}{2}} \left(\|a\|_\infty + \|b\|_\infty \right)$$

$$\leq \frac{1}{n+\frac{1}{2}} \left(\frac{1}{2\delta} + \frac{1}{4\delta^2} \cdot \pi \right) \xrightarrow{n \to \infty} 0$$



Recall $f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} + \frac{1}{2}(x - \pi)$

□

Theorem: $\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$, $x \in [0, 2\pi]$

↖ uniform convergence on $[0, 2\pi]$

Proof: For $\varepsilon > 0$, $x, x_0 \in [\varepsilon, 2\pi - \varepsilon]$: (use Lemma)

$$\int_{x_0}^{x} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k}\,dt = \int_{x_0}^{x} \frac{\pi - t}{2}\,dt = -\frac{(\pi - t)^2}{4} \Bigg|_{x_0}^{x} = -\frac{(x-\pi)^2}{4} + \underbrace{\frac{(x_0 - \pi)^2}{4}}_{C_0}$$

uniform convergence → ‖

$$\sum_{k=1}^{\infty} \int_{x_0}^{x} \frac{\sin(kt)}{k}\,dt = \sum_{k=1}^{\infty} -\frac{\cos(kt))}{k^2} \Bigg|_{x_0}^{x} = -\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} + C_1$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C \quad \leftarrow \text{calculate it!}$$

↘ still uniform convergence on $[\varepsilon, 2\pi - \varepsilon]$

We know more:

(1) $\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}$ uniformly convergent on $[0, 2\pi]$

by Weierstrass M-test since $\left| \frac{\cos(kx))}{k^2} \right| \leq \frac{1}{k^2}$

$\Rightarrow [0, 2\pi] \ni x \mapsto \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}$ continuous function

(2) $[0, 2\pi] \ni x \mapsto \frac{(x-\pi)^2}{4} + C$ continuous function

(3) $\sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$ for all $x \in (0, 2\pi)$

$\Rightarrow \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2} = \frac{(x-\pi)^2}{4} + C$ uniformly convergent on $[0, 2\pi]$

Find C : $\int_{0}^{2\pi} \sum_{k=1}^{\infty} \frac{\cos(kx))}{k^2}\,dx = \int_{0}^{2\pi} \left(\frac{(x-\pi)^2}{4} + C \right) dx = \underbrace{\frac{(x-\pi)^3}{12} \Bigg|_{0}^{2\pi}}_{\frac{\pi^3}{6}} + 2\pi \cdot C$

‖ uniform convergence

$$\sum_{k=1}^{\infty} \int_{0}^{2\pi} \frac{\cos(kx))}{k^2}\,dx = 0 \quad \Rightarrow \quad C = -\frac{\pi^2}{12}$$