The Bright Side of
Mathematics

Fourier Transform - Part 13

Theorem: $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with inner product $\langle f, g \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)} \cdot g(x)\, dx$

and ONS $(\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots)$ given by $e_k : x \mapsto e^{ikx}$.

For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ define: $\mathcal{F}_n(f) = \sum_{k=-n}^{n} e_k \underbrace{\langle e_k, f \rangle}_{c_k}$.

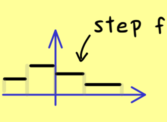
Then: $\| f - \mathcal{F}_n(f) \| \xrightarrow{n \to \infty} 0$ ← L^2-norm

(equivalent to Parseval's identity: $\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, f \rangle|^2$)

Fact: Continuous functions are dense in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$, which means:

For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ and $\varepsilon > 0$, there is a 2π-periodic continuous function $g: \mathbb{R} \to \mathbb{C}$ with $\| f - g \| < \varepsilon$. ← L^2-norm

Proposition: $\mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ is dense in $L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$.

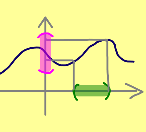


Proof: Let $\varepsilon > 0$, $f: [-\pi, \pi] \to \mathbb{C}$ square integrable.

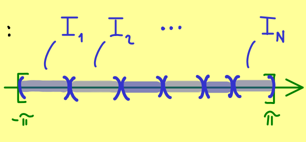
Then there is a continuous function $g: [-\pi,\pi] \to \mathbb{C}$ with $\| f - g \| < \varepsilon$.

$\overset{\text{domain compact}}{\Longrightarrow}$ g is uniformly continuous : for given $\varepsilon > 0$ there $\delta > 0$:

$$|x-y| < \delta \implies |g(x) - g(y)| < \varepsilon$$

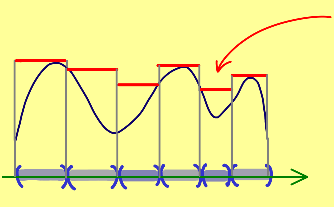


Decompose $[-\pi, \pi]$: $I_1, I_2, \dots, I_N$



$\text{length}(I_j) < \delta$

$c_j := \sup \{ g(x) \mid x \in \overline{I_j} \}$



define step function:

$h(x) = c_j$ for $x \in I_j$

We get: $|g(x) - h(x)| = |g(x) - g(y)|$ for $y \in \overline{I_j}$ (with $x \in I_j$); $< \varepsilon$ because $|x-y| < \delta$

In total: $\|f - h\| \le \underbrace{\|f - g\|}_{< \varepsilon} + \underbrace{\|g - h\|}_{= \left(\int_{-\pi}^{\pi} \underbrace{|g(x) - h(x)|^2}_{< \varepsilon} \right)^{1/2}} < C \cdot \varepsilon$ (constant) □

Theorem (see above): For $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$: $\| f - \mathcal{F}_n(f) \| \xrightarrow{n \to \infty} 0$

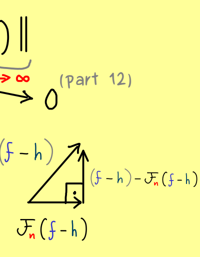
Proof: Let $\varepsilon > 0$, $f \in L^2_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$. Choose $h \in \mathcal{S}_{2\pi\text{-per}}(\mathbb{R},\mathbb{C})$ with $\|f - h\| < \varepsilon$.

Then: $\| f - \mathcal{F}_n(f) \| = \| f + h - h - \mathcal{F}_n(f) + \mathcal{F}_n(h) - \mathcal{F}_n(h) \|$

$$\le \underbrace{\| (f-h) - \mathcal{F}_n(f-h) \|}_{\le \|(f-h)\| < \varepsilon} + \underbrace{\| h - \mathcal{F}_n(h) \|}_{\xrightarrow{h \to \infty} 0 \text{ (part 12)}}$$

Pythagorean theorem:

$$\|(f-h) - \mathcal{F}_n(f-h)\|^2 + \|\mathcal{F}_n(f-h)\|^2 = \|(f-h)\|^2$$



$$\implies \lim_{n \to \infty} \| f - \mathcal{F}_n(f) \| = 0$$ □