The Bright Side of Mathematics

Fourier Transform - Part 15

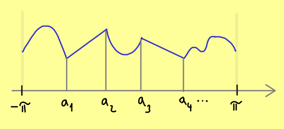
Theorem: $f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic continuous function

and piecewise C^1-function :

there are finitely many points $(a_1, a_2, \dots, a_m)$

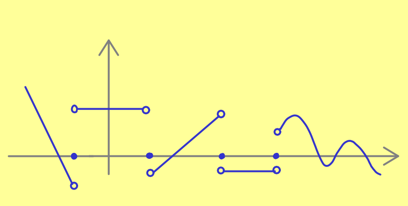
inside the interval $[-\pi, \pi]$ such that: $f|_{[a_j, a_{j+1}]} \in C^1$

for all $j \in \{0,1,\dots,m\}$, $a_0 := -\pi$, $a_{m+1} := \pi$



Then: $\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ uniformly.

Proof: Consider the derivative function: $\tilde{f}(x) := \begin{cases} 0 , & x \in \{a_0, a_1, \dots, a_{m+1}\} \\ f'(x) , & \text{else} \end{cases}$



piecewise continuous function $\in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

Parseval's identity: $\|\tilde{f}\|^2 = \sum_{k=-\infty}^{\infty} |\langle e_k, \tilde{f} \rangle|^2 < \infty$

What about the Fourier coefficients of f ? $(k \neq 0)$

$$c_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{e^{-ikx}}_{u'} \underbrace{f(x)}_{v} \, dx = \frac{1}{2\pi} \left(u \cdot v \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u v' \, dx \right)$$

$u = \frac{1}{-ik} e^{-ikx}$ integration by parts

$$= \frac{1}{2\pi} \left(0 + \frac{1}{ik} \int_{-\pi}^{\pi} e^{-ikx} \tilde{f}(x) \, dx \right) = \frac{1}{ik} \langle e_k, \tilde{f} \rangle$$

General inequality for real numbers: $x \cdot y \leq \frac{x^2 + y^2}{2}$

$$|c_k| = \frac{1}{k} |\langle e_k, \tilde{f} \rangle| \leq \frac{1}{2} \left(\frac{1}{k^2} + |\langle e_k, \tilde{f} \rangle|^2 \right)$$

$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |c_k| \leq \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k^2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} |\langle e_k, \tilde{f} \rangle|^2 < \infty$$

$$\mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} \underbrace{e^{ikx} \cdot c_k}_{f_k(x)} \quad \text{with } |f_k(x)| \leq M_k =: |c_k| , \quad \sum_{k=-\infty}^{\infty} M_k < \infty$$

Weierstrass M-Test $\Longrightarrow$ $\sum_{k=-\infty}^{\infty} f_k$ uniformly convergent to a continuous function $h: [-\pi, \pi] \longrightarrow \mathbb{C}$

Status quo: $\| \mathcal{F}_n(f) - h \|_\infty \xrightarrow{n \to \infty} 0$, $\| \mathcal{F}_n(f) - f \|_{L^2} \xrightarrow{n \to \infty} 0$

More estimates: $\| f - h \|_{L^2} \leq \| f - \mathcal{F}_n(f) \|_{L^2} + \underbrace{\| \mathcal{F}_n(f) - h \|_{L^2}}_{\leq \| \mathcal{F}_n(f) - h \|_\infty} \xrightarrow{n \to \infty} 0$

Hence: $\| f - h \|_{L^2} = 0 \overset{\text{continuous functions}}{\Longrightarrow} f = h$

Conclusion: $\| \mathcal{F}_n(f) - f \|_\infty \xrightarrow{n \to \infty} 0$ (uniform convergence of the Fourier series)

$\square$