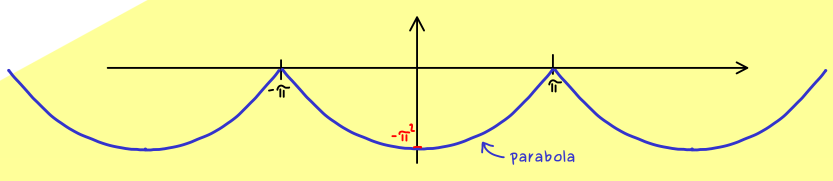
The Bright Side of
Mathematics

Fourier Transform - Part 16



$\Rightarrow$ continuous + piecewise C^1-function

Example: $f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic with $f(x) = x^2 - \pi^2$ for $x \in [-\pi, \pi]$.

Let's calculate the Fourier coefficients: $c_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 - \pi^2)\, dx = \frac{1}{2\pi} \left(\frac{1}{3} x^3 - \pi^2 x \right) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \cdot 2 \cdot \left(\frac{1}{3} \pi^3 - \pi^3 \right) = \underline{-\frac{2}{3} \pi^2}$$

For $k \neq 0$: $c_k = \frac{1}{ik} \langle e_k, f' \rangle$ (integration by parts, see part 15)

$$= \frac{1}{2\pi i k} \int_{-\pi}^{\pi} \underbrace{e^{-ikx}}_{u'} \underbrace{2 \cdot x}_{v} \, dx \quad \text{(integration by parts)}$$

$\rightsquigarrow u = \frac{1}{-ik} e^{-ikx}$, $v' = 2$

$$= \frac{1}{2\pi i k} \left(-\frac{1}{ik} e^{-ikx} \cdot 2 \cdot x \Big|_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} \left(-\frac{1}{ik} e^{-ikx} \right) \cdot 2\, dx}_{= 0} \right)$$

$$= \frac{1}{\pi \cdot k^2} \left(\underbrace{e^{-ik\pi}}_{= (-1)^k} \pi - \underbrace{e^{ik\pi}}_{= (-1)^k} (-\pi) \right)$$

$$= \frac{2 \cdot (-1)^k}{k^2}$$

Fourier series: $x \in [-\pi, \pi]$

$$x^2 - \pi^2 = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = -\frac{2}{3}\pi^2 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{2 \cdot (-1)^k}{k^2} \underbrace{e^{ikx}}_{= \cos(kx) + i \cdot \sin(kx)}$$

$$= -\frac{2}{3}\pi^2 + 2 \cdot \sum_{k=1}^{\infty} \frac{2 \cdot (-1)^k}{k^2} \cos(kx)$$

For all $x \in [-\pi, \pi]$: $x^2 - \frac{1}{3}\pi^2 = \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k \cos(kx)$ ← uniform convergence!

In particular for $x = 0$: $-\frac{1}{3}\pi^2 = \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{1}{12}\pi^2}$$

Parseval's identity:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \| f \|_{L^2}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 - \pi^2)^2\, dx = \frac{8}{15} \pi^4$$

$$\| \quad |c_0|^2 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left| \frac{2 \cdot (-1)^k}{k^2} \right|^2$$

$$\| \quad \frac{4}{9} \cdot \pi^4 + 2 \cdot \sum_{k=1}^{\infty} \frac{4}{k^4} \qquad \Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}}$$