The Bright Side of
Mathematics

Fourier Transform - Part 17

$f: \mathbb{R} \longrightarrow \mathbb{C}$ 2π-periodic

$f \in \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$

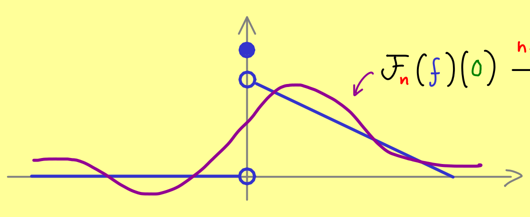
continuous + piecewise C^1-function

$\Longrightarrow$

$\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ (in L^2-norm)

$\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ (pointwisely)

$\mathcal{F}_n(f) \xrightarrow{n \to \infty} f$ (uniformly)



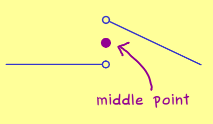
$\mathcal{F}_n(f)(0) \xrightarrow{n \to \infty} ?$

Theorem: $f \in \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$, $\hat{x} \in [-\pi, \pi]$ with:

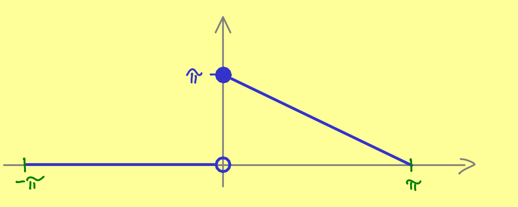
$f(\hat{x}^-) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} - \varepsilon)$ exists, $\lim_{\substack{h \to 0 \\ h < 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h}$ exists

$f(\hat{x}^+) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} + \varepsilon)$ exists, $\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h}$ exists

Then: $\mathcal{F}_n(f)(\hat{x}) \xrightarrow{n \to \infty} \frac{1}{2}\left(f(\hat{x}^+) + f(\hat{x}^-) \right)$



Example:



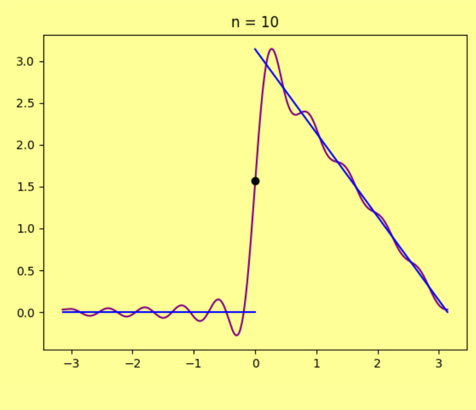
$$f(x) = \begin{cases} 0 & , \ x \in [-\pi, 0) \\ \pi - x & , \ x \in [0, \pi) \end{cases}$$

Fourier coefficients: $c_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_0^\pi e^{-ikx} (\pi - x) \, dx$

$$= \begin{cases} \frac{\pi}{4} & , \ k = 0 \\ \frac{1}{2\pi} \cdot \left(-\frac{1}{k^2}\left((-1)^k - 1\right) - i \frac{\pi}{k} \right) & , \ k \neq 0 \end{cases}$$

Fourier series: $\mathcal{F}_n(f)(x) = \frac{\pi}{4} + \sum_{\substack{k=-n \\ k \neq 0}}^{n} c_k \cdot e^{ikx}$

$a_k = c_k + c_{-k}$
$b_k = i \cdot (c_k - c_{-k})$



$$= \frac{\pi}{4} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$$

with $a_k = \frac{1}{\pi} \cdot \frac{1}{k^2}\left(1 - (-1)^k\right)$, $b_k = \frac{1}{k}$

$\mathcal{F}_n(f)(0) \xrightarrow{n \to \infty} \frac{\pi}{2}$

$$\Longrightarrow \quad \frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{1}{\pi} \frac{1-(-1)^k}{k^2}$$

$$\Longrightarrow \quad \frac{\pi^2}{4} = \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k^2}$$