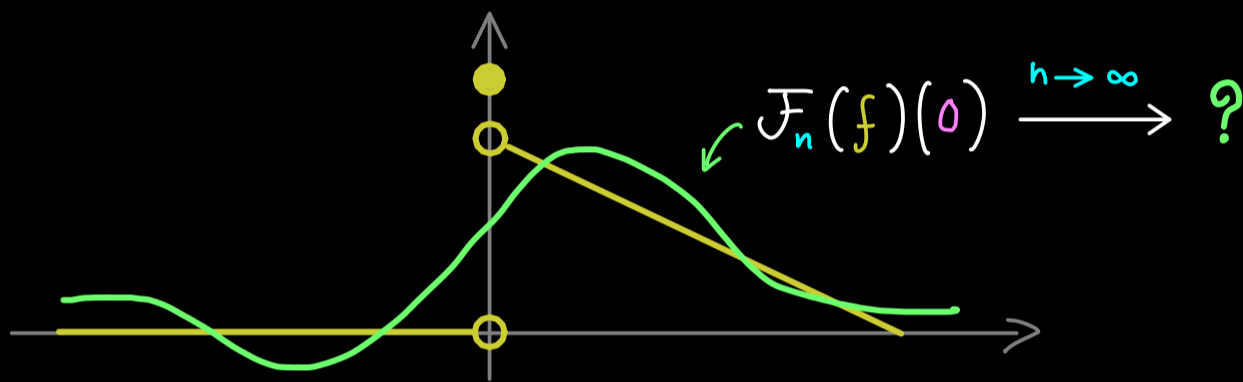


## Fourier Transform - Part 17

$$\begin{array}{l}
 f: \mathbb{R} \rightarrow \mathbb{C} \quad 2\pi\text{-periodic} \\
 f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C}) \\
 \text{continuous + piecewise } C^1\text{-function}
 \end{array}
 \implies
 \begin{array}{l}
 \mathcal{F}_n(f) \xrightarrow{h \rightarrow \infty} f \quad (\text{in } L^2\text{-norm}) \\
 \mathcal{F}_n(f) \xrightarrow{h \rightarrow \infty} f \quad (\text{pointwisely}) \\
 \mathcal{F}_n(f) \xrightarrow{h \rightarrow \infty} f \quad (\text{uniformly})
 \end{array}$$



Theorem:  $f \in L^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$ ,  $\hat{x} \in [-\pi, \pi]$  with:

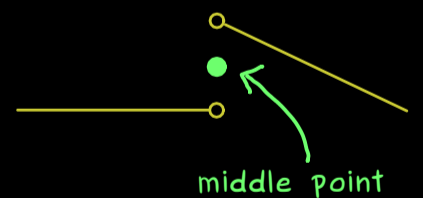
$$f(\hat{x}^-) := \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} f(\hat{x} - \epsilon) \text{ exists,}$$

$$\lim_{\substack{h \rightarrow 0 \\ h < 0}} \frac{f(\hat{x} + h) - f(\hat{x})}{h} \text{ exists}$$

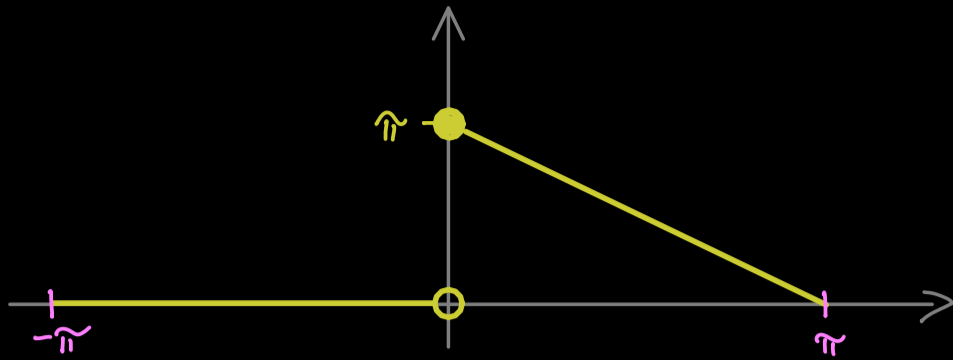
$$f(\hat{x}^+) := \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} f(\hat{x} + \epsilon) \text{ exists,}$$

$$\lim_{\substack{h \rightarrow 0 \\ h > 0}} \frac{f(\hat{x} + h) - f(\hat{x})}{h} \text{ exists}$$

Then:  $\mathcal{F}_n(f)(\hat{x}) \xrightarrow{h \rightarrow \infty} \frac{1}{2} (f(\hat{x}^+) + f(\hat{x}^-))$



Example:



$$f(x) = \begin{cases} 0 & , x \in [-\pi, 0) \\ \pi - x & , x \in [0, \pi) \end{cases}$$

Fourier coefficients:  $C_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_0^{\pi} e^{-ikx} (\pi - x) dx$

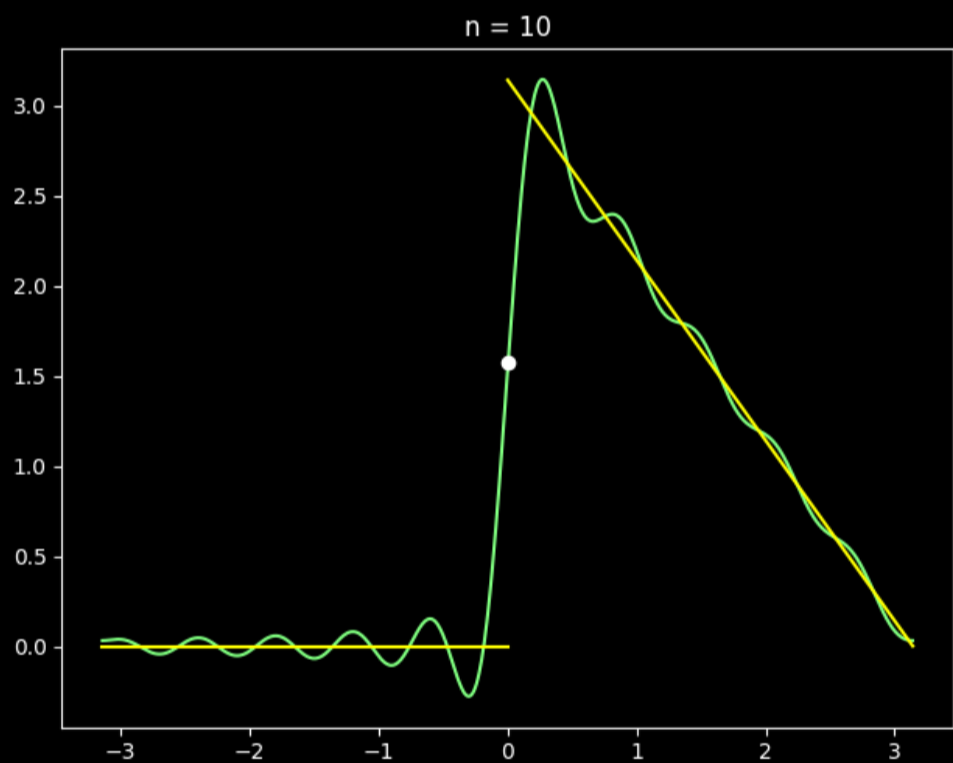
$$= \begin{cases} \frac{\pi}{4} & , k=0 \\ \frac{1}{2\pi} \cdot \left( -\frac{1}{k^2}((-1)^k - 1) - i \frac{\pi}{k} \right) & , k \neq 0 \end{cases}$$

Fourier series:

$$\mathcal{F}_n(f)(x) = \frac{\pi}{4} + \sum_{\substack{k=-n \\ k \neq 0}}^n C_k \cdot e^{ikx}$$

$$a_k = C_k + C_{-k}$$

$$b_k = i \cdot (C_k - C_{-k})$$



$$= \frac{\pi}{4} + \sum_{k=1}^n a_k \cos(kx) + \sum_{k=1}^n b_k \sin(kx)$$

$a_k = \frac{1}{\pi} \cdot \frac{1}{k^2} (1 - (-1)^k)$        $b_k = \frac{1}{k}$

$$\mathcal{F}_n(f)(0) \xrightarrow{n \rightarrow \infty} \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{1}{\pi} \frac{1 - (-1)^k}{k^2}$$

$$\Rightarrow \frac{\pi^2}{4} = \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k^2}$$