The Bright Side of
Mathematics

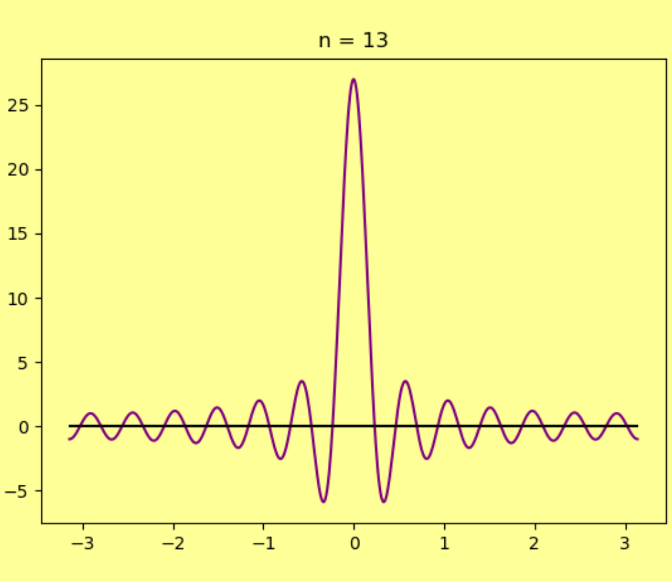
Fourier Transform - Part 18

Definition: The continuous function $\mathcal{D}_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, given by

$$\mathcal{D}_n(x) = \sum_{k=-n}^{n} e^{ikx} \underset{\text{part 11}}{=} 1 + 2 \cdot \sum_{k=1}^{n} \cos(kx) \underset{\text{part 11}}{=} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

$2\pi \cdot m$ zeros for $m \in \mathbb{Z}$

is called the Dirichlet kernel.



2π-periodic

For Fourier series:

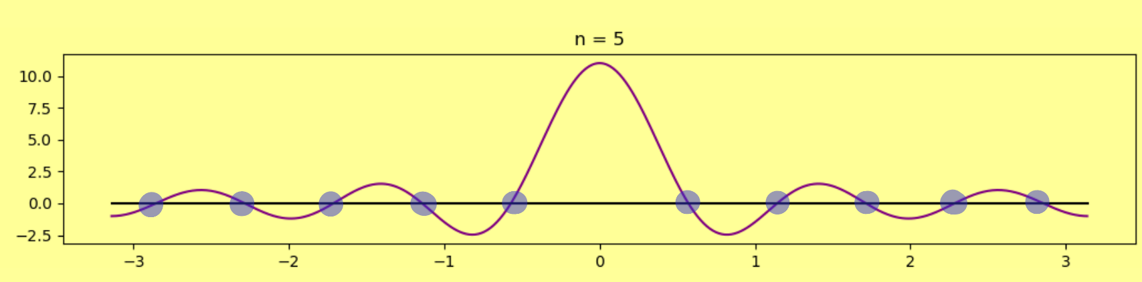
$$\mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} c_k \cdot e^{ikx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{\pi}^{\pi} e^{-iky} f(y)\, dy \right) \cdot e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{ik(x-y)}\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \mathcal{D}_n(\underbrace{x-y}_{z})\, dy$$

$$= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-z)\, \mathcal{D}_n(z)\, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{D}_n(z)\, f(x-z)\, dz$$

$$= \left\langle \mathcal{D}_n , f(x-\cdot) \right\rangle = \frac{1}{2\pi} (\mathcal{D}_n * f)(x) \quad \text{(convolution)}$$

Properties: (1) $\mathcal{D}_n$ has exactly $2n$ zeros inside the interval $[-\pi, \pi]$

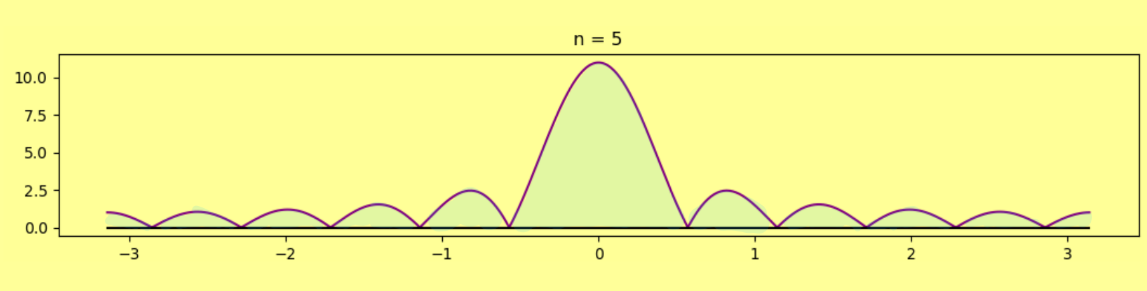


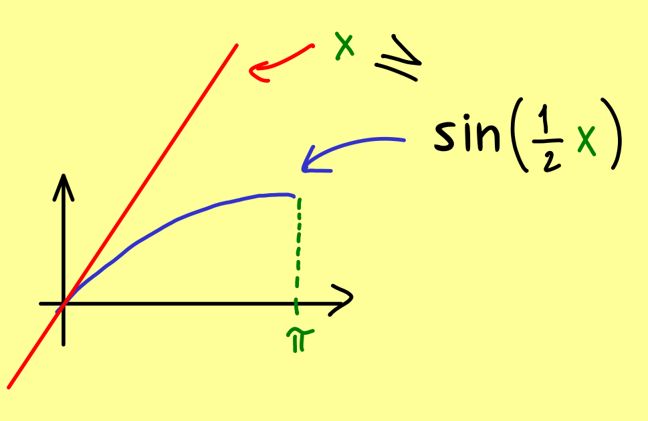
$$\frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

(2) $$\int_{-\pi}^{\pi} \mathcal{D}_n(x)\, dx = \int_{-\pi}^{\pi} \left(1 + e^{ix} + e^{-ix} + e^{2ix} + e^{-2ix} + \dots + e^{nix} + e^{-nix}\right) dx$$

$$= 2\pi \qquad \Rightarrow \left\langle \mathcal{D}_n , 1 \right\rangle = 1$$

(3) $$\int_{-\pi}^{\pi} |\mathcal{D}_n(x)|\, dx \xrightarrow{n \to \infty} \infty$$



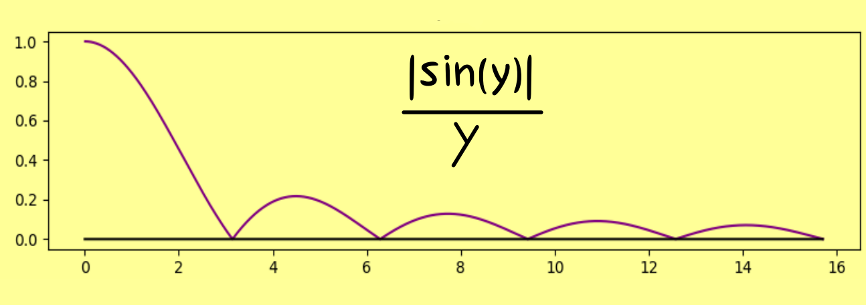


Proof of (3): $$|\mathcal{D}_n(x)| = \frac{|\sin((n+\frac{1}{2})x)|}{|\sin(\frac{1}{2}x)|}$$

$$\geq \frac{|\sin((n+\frac{1}{2})x)|}{x} \quad \text{for all } x > 0$$

$$\int_{-\pi}^{\pi} |\mathcal{D}_n(x)|\, dx = 2 \cdot \int_{0}^{\pi} |\mathcal{D}_n(x)|\, dx \geq 2 \cdot \int_{0}^{\pi} \frac{|\sin(\overbrace{(n+\frac{1}{2})x}^{y})|}{x}\, dx$$

$$= 2 \cdot \int_{0}^{(n+\frac{1}{2})\pi} \frac{|\sin(y)|}{y}\, dy \geq 2 \cdot \int_{0}^{n \cdot \pi} \frac{|\sin(y)|}{y}\, dy$$



$$= 2 \cdot \sum_{k=1}^{n} \int_{(k-1)\cdot\pi}^{k\cdot\pi} \frac{|\sin(y)|}{y}\, dy$$

maximal $k \cdot \pi$

$$\geq 2 \cdot \sum_{k=1}^{n} \int_{(k-1)\cdot\pi}^{k\cdot\pi} \frac{|\sin(y)|}{k \cdot \pi}\, dy$$

$$= 2 \cdot \sum_{k=1}^{n} \frac{1}{k\pi} \underbrace{\int_{(k-1)\cdot\pi}^{k\cdot\pi} |\sin(y)|\, dy}_{=1} = \text{const} \cdot \sum_{k=1}^{n} \frac{1}{k}$$

$$\xrightarrow{n \to \infty} \infty$$