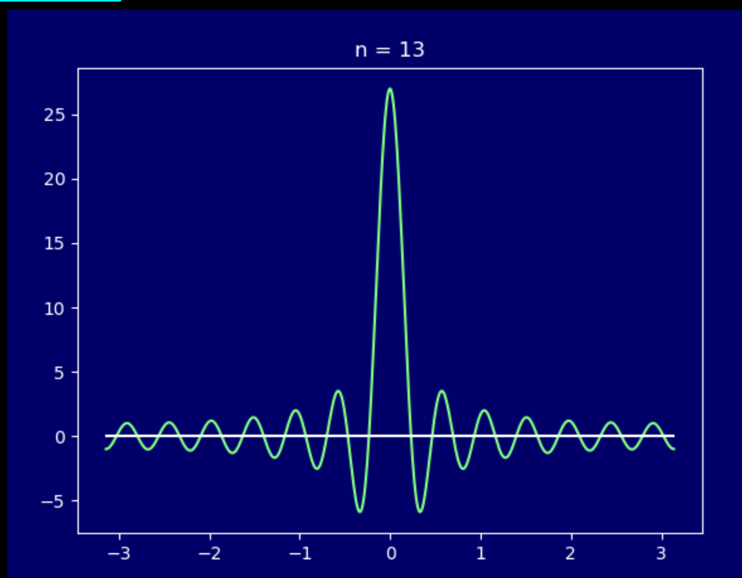
# Fourier Transform - Part 18

**Definition:** The continuous function $\mathcal{D}_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, given by

$$\mathcal{D}_n(x) = \sum_{k=-n}^{n} e^{ikx} \underset{\text{part 11}}{=} 1 + 2 \cdot \sum_{k=1}^{n} \cos(kx) \underset{\text{part 11}}{=} \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

$2\pi \cdot m$ zeros for $m \in \mathbb{Z}$

is called the Dirichlet kernel.




$2\pi$-periodic

**For Fourier series:**

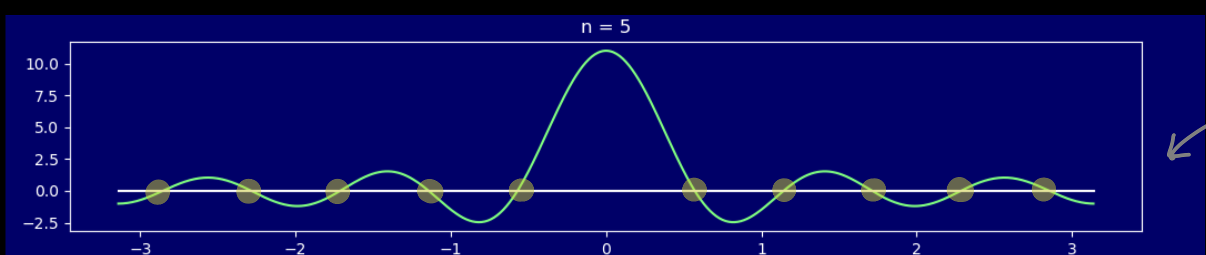
$$\mathcal{F}_n(f)(x) = \sum_{k=-n}^{n} c_k \cdot e^{ikx} = \sum_{k=-n}^{n} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iky} f(y)\, dy \right) \cdot e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{ik(x-y)}\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \mathcal{D}_n(\underbrace{x-y}_{z})\, dy$$

$$= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-z)\, \mathcal{D}_n(z)\, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{D}_n(z)\, f(x-z)\, dz$$

$$= \left\langle \mathcal{D}_n, f(x-\cdot) \right\rangle = \frac{1}{2\pi} \left( \mathcal{D}_n * f \right)(x) \qquad \text{(convolution)}$$

**Properties:** (1) $\mathcal{D}_n$ has exactly $2n$ zeros inside the interval $[-\pi, \pi]$

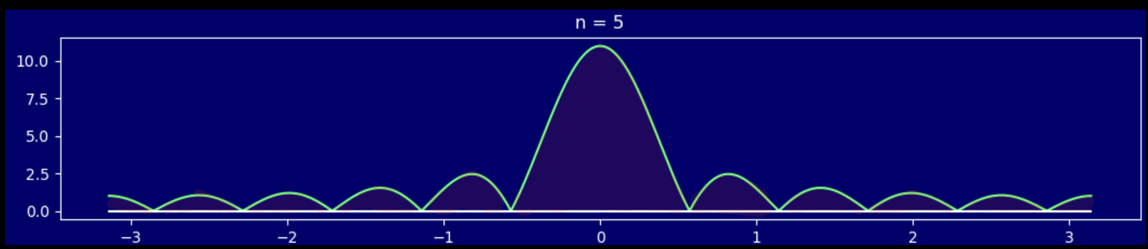


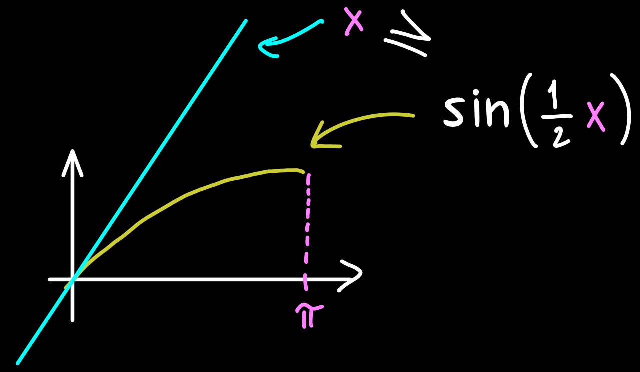

$$\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

$$(2)\quad \int_{-\pi}^{\pi} D_n(x)\,dx = \int_{-\pi}^{\pi}\left(1 + e^{ix} + e^{-ix} + e^{2ix} + e^{-2ix} + \dots + e^{nix} + e^{-nix}\right)dx$$

$$= 2\pi \qquad \Rightarrow \quad \langle D_n, 1 \rangle = 1$$

$$(3)\quad \int_{-\pi}^{\pi} |D_n(x)|\,dx \xrightarrow{n\to\infty} \infty$$




Proof of (3):

$$|D_n(x)| = \frac{|\sin((n+\frac{1}{2})x)|}{|\sin(\frac{1}{2}x)|}$$

$$\geq \frac{|\sin((n+\frac{1}{2})x)|}{x} \qquad \text{for all } x > 0$$

$$\int_{-\pi}^{\pi} |D_n(x)|\,dx = 2\cdot\int_0^{\pi} |D_n(x)|\,dx \geq 2\cdot\int_0^{\pi} \frac{|\sin(\overbrace{(n+\frac{1}{2})x}^{y})|}{x}\,dx$$

$$= 2\cdot\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin(y)|}{y}\,dy \geq 2\cdot\int_0^{n\cdot\pi} \frac{|\sin(y)|}{y}\,dy$$

$$\frac{|\sin(y)|}{y}$$

$$= 2\cdot\sum_{k=1}^{n} \int_{(k-1)\cdot\pi}^{k\cdot\pi} \frac{|\sin(y)|}{y}\,dy$$

maximal $k\cdot\pi$

$$\geq 2\cdot\sum_{k=1}^{n} \int_{(k-1)\cdot\pi}^{k\cdot\pi} \frac{|\sin(y)|}{k\cdot\pi}\,dy$$

$$= 2\cdot\sum_{k=1}^{n}\frac{1}{k\pi} \underbrace{\int_{(k-1)\cdot\pi}^{k\cdot\pi} |\sin(y)|\,dy}_{=1} = \text{const}\cdot\sum_{k=1}^{n}\frac{1}{k}$$

$$\xrightarrow{n\to\infty} \infty$$