

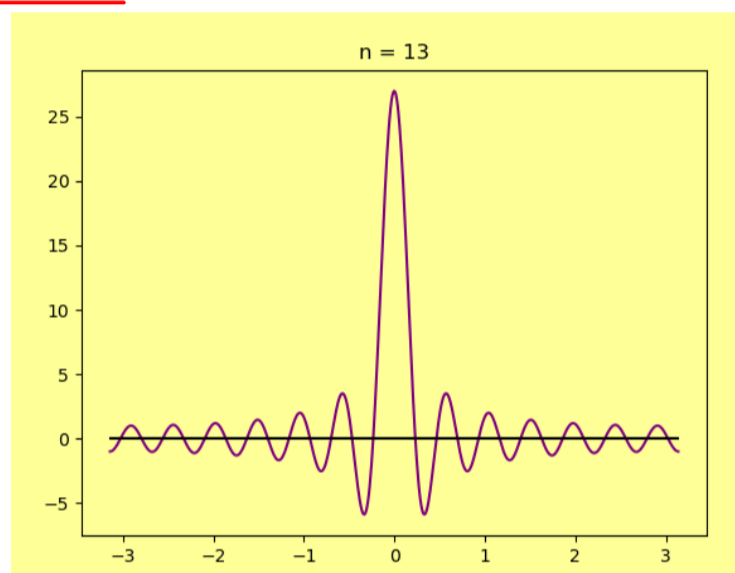
Fourier Transform - Part 18

Definition: The continuous function $\mathcal{D}_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, given by

$$\mathcal{D}_n(x) = \sum_{k=-n}^n e^{ikx} \stackrel{\text{part 11}}{=} 1 + 2 \cdot \sum_{k=1}^n \cos(kx) \stackrel{\text{part 11}}{=} \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

is called the Dirichlet kernel.

$2\pi \cdot m$ zeros
for $m \in \mathbb{Z}$

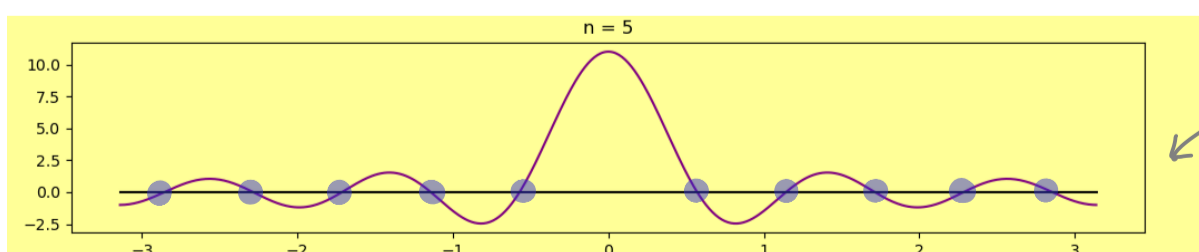


2π -periodic

For Fourier series:

$$\begin{aligned} \mathcal{F}_n(f)(x) &= \sum_{k=-n}^n c_k \cdot e^{ikx} = \sum_{k=-n}^n \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iky} f(y) dy \right) \cdot e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-n}^n e^{ik(x-y)} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \mathcal{D}_n(\underbrace{x-y}_z) dy \\ &= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-z) \mathcal{D}_n(z) dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{D}_n(z) f(x-z) dz \\ &= \langle \mathcal{D}_n, f(x-\cdot) \rangle = \frac{1}{2\pi} (\mathcal{D}_n * f)(x) \quad (\text{convolution}) \end{aligned}$$

Properties: (1) \mathcal{D}_n has exactly $2n$ zeros inside the interval $[-\pi, \pi]$

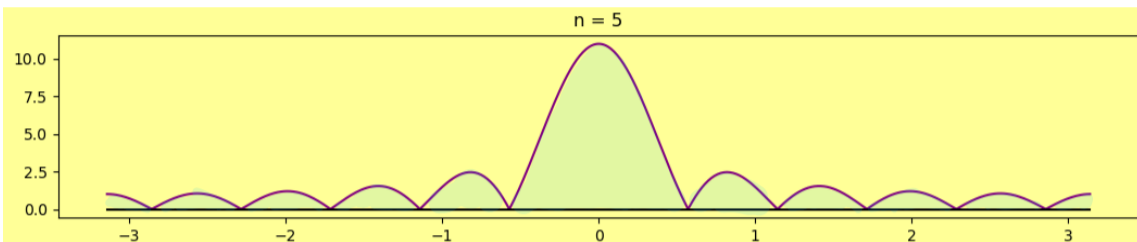


$$\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

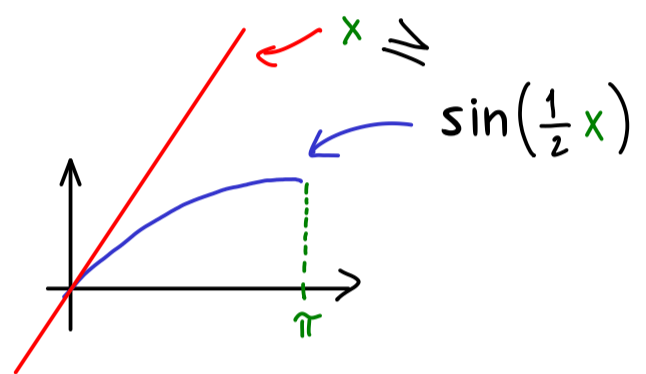
$$(2) \int_{-\pi}^{\pi} \mathcal{D}_n(x) dx = \int_{-\pi}^{\pi} (1 + e^{ix} + e^{-ix} + e^{2ix} + e^{-2ix} + \dots + e^{nix} + e^{-nix}) dx$$

$$= 2\pi \quad \Rightarrow \quad \langle \mathcal{D}_n, 1 \rangle = 1$$

$$(3) \int_{-\pi}^{\pi} |\mathcal{D}_n(x)| dx \xrightarrow{h \rightarrow \infty} \infty$$



Proof of (3): $|\mathcal{D}_n(x)| = \frac{|\sin((n+\frac{1}{2})x)|}{|\sin(\frac{1}{2}x)|}$



$$\geq \frac{|\sin((n+\frac{1}{2})x)|}{x} \quad \text{for all } x > 0$$

$$\int_{-\pi}^{\pi} |\mathcal{D}_n(x)| dx = 2 \cdot \int_0^{\pi} |\mathcal{D}_n(x)| dx \geq 2 \cdot \int_0^{\pi} \frac{|\sin((n+\frac{1}{2})x)|}{x} dx$$

$$= 2 \cdot \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin(y)|}{y} dy \geq 2 \cdot \int_0^{n\pi} \frac{|\sin(y)|}{y} dy$$

$$= 2 \cdot \sum_{k=1}^h \int_{(k-1)\pi}^{k\pi} \frac{|\sin(y)|}{y} dy$$

maximal $k\pi$

$$\geq 2 \cdot \sum_{k=1}^h \int_{(k-1)\pi}^{k\pi} \frac{|\sin(y)|}{k\pi} dy$$

$$= 2 \cdot \sum_{k=1}^h \frac{1}{k\pi} \underbrace{\int_{(k-1)\pi}^{k\pi} |\sin(y)| dy}_{=1} = \text{const} \cdot \sum_{k=1}^h \frac{1}{k}$$

$$\xrightarrow{h \rightarrow \infty} \infty$$

$$\frac{|\sin(y)|}{y}$$