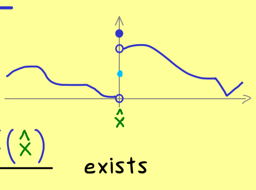
The Bright Side of Mathematics

Fourier Transform - Part 19

Theorem: $f \in \mathcal{L}^2_{2\pi\text{-per}}(\mathbb{R}, \mathbb{C})$, $\hat{x} \in [-\pi, \pi]$ with:



$$f(\hat{x}^-) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} - \varepsilon) \text{ exists}, \quad \lim_{\substack{h \to 0 \\ h < 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} \text{ exists}$$

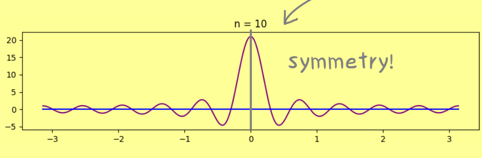
$$f(\hat{x}^+) := \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(\hat{x} + \varepsilon) \text{ exists}, \quad \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} \text{ exists}$$

Then:

$$\mathcal{F}_n(f)(\hat{x}) \xrightarrow{n \to \infty} \frac{1}{2}\Big(f(\hat{x}^+) + f(\hat{x}^-)\Big) =: M$$

Proof: Dirichlet kernel: $\mathcal{D}_n(x) = \dfrac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{1}{2}x\right)}$ gives $\mathcal{F}_n(f)(\hat{x}) = \langle \mathcal{D}_n, f(\hat{x} - \cdot) \rangle$

and $\langle \mathcal{D}_n, M \rangle = M$



Use symmetry:

$$\langle \mathcal{D}_n, f(\hat{x} - \cdot) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{D}_n(x) f(\hat{x} - x)\, dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0} \mathcal{D}_n(x) f(\hat{x} \underbrace{-x}_{y})\, dx + \int_{0}^{\pi} \mathcal{D}_n(x) f(\hat{x} - x)\, dx \right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{\pi} \mathcal{D}_n(y) f(\hat{x} + y)\, dy + \int_{0}^{\pi} \mathcal{D}_n(x) f(\hat{x} - x)\, dx \right)$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} \mathcal{D}_n(y) \Big(f(\hat{x} + y) + f(\hat{x} - y) \Big)\, dy$$

Pointwise limit:

$$\mathcal{F}_n(f)(\hat{x}) - M = \langle \mathcal{D}_n, f(\hat{x} - \cdot) \rangle - \langle \mathcal{D}_n, M \rangle$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} \mathcal{D}_n(y) \Big(f(\hat{x}+y) + f(\hat{x}-y) \Big)\, dy - \frac{1}{2\pi} \int_0^{\pi} \mathcal{D}_n(y) \underbrace{2 \cdot M}_{= f(\hat{x}^+) + f(\hat{x}^-)}\, dy$$

$$= \frac{1}{2\pi} \int_0^{\pi} \mathcal{D}_n(y) \Big(f(\hat{x}+y) - f(\hat{x}^+) + f(\hat{x}-y) - f(\hat{x}^-) \Big)\, dy$$

$$= \frac{1}{2\pi} \int_0^{\pi} \sin\left((n+\tfrac{1}{2})y\right) \underbrace{\frac{f(\hat{x}+y) - f(\hat{x}^+) + f(\hat{x}-y) - f(\hat{x}^-)}{\sin\left(\frac{1}{2}y\right)}}_{g(y)}\, dy$$

In the case that $g \in L^2_{2\pi\text{-per}}$, we get:

$$\frac{1}{2\pi} \int_0^{\pi} \underbrace{\sin\left((n+\tfrac{1}{2})y\right)}_{\frac{1}{2i}\cdot\left(e^{iny} e^{i\frac{1}{2}y} - e^{-iny} e^{-i\frac{1}{2}y}\right)} g(y)\, dy$$

$$= \langle e_{-n}, g_1 \rangle + \langle e_n, g_2 \rangle \xrightarrow[\text{part 8}]{n \to \infty} 0$$

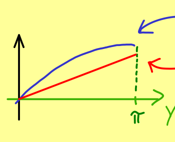
(Bessel's inequality)

g_1, g_2: $L^2_{2\pi\text{-per}}$-functions

Show that $g \in L^2_{2\pi\text{-per}}$:

$$g(y) = \begin{cases} \dfrac{f(\hat{x}+y) - f(\hat{x}^+)}{\sin\left(\frac{1}{2}y\right)} + \dfrac{f(\hat{x}-y) - f(\hat{x}^-)}{\sin\left(\frac{1}{2}y\right)}, & y \in (0, \pi) \\ 0, & y \in [-\pi, 0] \end{cases}$$

Does $g(y)$ explode for $y \to 0^+$?



$$\sin\left(\tfrac{1}{2}y\right) \geq \tfrac{1}{4}y \implies \left| \frac{f(\hat{x}+y) - f(\hat{x}^+)}{\sin\left(\frac{1}{2}y\right)} \right| \leq 4 \cdot \left| \frac{f(\hat{x}+y) - f(\hat{x}^+)}{y} \right| \xrightarrow{y \to 0^+} 4 \cdot |C^+|$$

because $\displaystyle \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} =: C^+$

and

$$\left| \frac{f(\hat{x}-y) - f(\hat{x}^-)}{\sin\left(\frac{1}{2}y\right)} \right| \leq 4 \cdot \left| \frac{f(\hat{x}-y) - f(\hat{x}^-)}{y} \right| \xrightarrow{y \to 0^+} 4 \cdot |C^-|$$

because $\displaystyle \lim_{\substack{h \to 0 \\ h < 0}} \frac{f(\hat{x}+h) - f(\hat{x})}{h} =: C^-$ □