## Exercise 1. A new metric

*Let $d$ be a metric on the set $X$. Let $a$ be a fixed point of $X$. Then we define $d_a : X \times X \to \mathbb{R}$:*

$$d_a(x,y) := \begin{cases} d(x,a) + d(y,a), & \text{for } x \neq y, \\ 0, & \text{else.} \end{cases}$$

*(a) Show that $(X, d_a)$ is a metric space.*

*(b) Now let $X = \mathbb{C}$ with the standard metric and $a = 1$. Give the open ball*

$$B_r(0) := \{z \in \mathbb{C}, d_1(z,0) \leq r\}$$

*for each $r \geq 0$.*

## Exercise 2. Sets with topological notions

*Let $(X,d)$ be a metric space and $A, B \subseteq X$. Let us use the definition of $\partial A$ from the video and define $A^\circ := A \setminus \partial A$ and $\overline{A} := A \cup \partial A$. Show the following statements:*

*(a) $\partial A = \overline{A} \setminus A^\circ$*

*(b) $\partial A = \partial(A^c)$ where $A^c$ is the complement of $A$.*

*(c) $(\overline{A})^c = (A^c)^\circ$ and $\overline{A^c} = (A^\circ)^c$*

*(d) $(A \cap B)^\circ = A^\circ \cap B^\circ$*

*(e) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. Give also an explicit example where we have $\neq$ in the relation.*

*(f) $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

*(g) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Give also an explicit example where we have $\neq$ in the relation.*

*(h) $\partial(A \cup B) \subseteq \partial A \cup \partial B$. Give also an explicit example where we have $\neq$ in the relation.*

*(i) $\partial(A \cap B) \subseteq \partial A \cap \partial B$. Give also an explicit example where we have $\neq$ in the relation.*

*(j) $\partial A \supseteq \partial(\overline{A})$. Give also an explicit example where we have $\neq$ in the relation.*

## Exercise 3. Balls in metric spaces

*Let $(X,d)$ be a metric space, $x \in X$, and $\varepsilon > 0$. Show the following:*

*(a) $B_\varepsilon(x) := \{y \in X \mid d(x,y) < \varepsilon\}$ is an open set.*

*(b) $\widetilde{B}_\varepsilon(x) := \{y \in X \mid d(x,y) \leq \varepsilon\}$ is a closed set.*

*(c) $S_\varepsilon(x) := \{y \in X \mid d(x,y) = \varepsilon\}$ is a closed set.*

*Give also an explicit example where $\overline{B_\varepsilon(x)} \neq \widetilde{B}_\varepsilon(x)$.*