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ON STEADY

The Bright Side of Mathematics

Functional analysis – part 17

Arzelà-Ascoli theorem

Example: (a) $(X, \|\cdot\|)$ normed space with $\dim(X) < \infty$ (always Banach space)

$A \subseteq X$: A compact $\Longleftrightarrow$ A closed + bounded

(b) $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ for $p \in [1, \infty)$ (Banach space)

$A := \{ x \in \ell^p(\mathbb{N}) \mid \|x\|_p \leq 1 \}$ closed + bounded

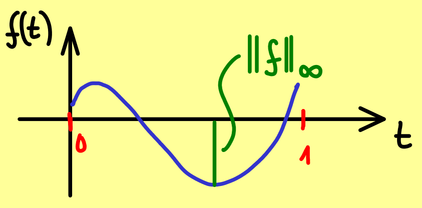
$$\left.\begin{array}{l} e_1 := (1, 0, 0, 0, \dots) \in A \\ e_2 := (0, 1, 0, 0, \dots) \in A \\ e_3 := (0, 0, 1, 0, \dots) \in A \\ \vdots \qquad\qquad \vdots \end{array}\right\} \quad (e_n)_{n \in \mathbb{N}} \subseteq A$$

$\|e_n - e_m\|_p \overset{n \neq m}{=} \sqrt[p]{|1|^p + |-1|^p} = \sqrt[p]{2}$

$\Rightarrow$ no convergent subsequence

Continuous functions: $(C([0,1]), \|\cdot\|_\infty)$, $\|f\|_\infty := \sup\{ |f(t)| \mid t \in [0,1] \}$

↳ Banach space



f is called uniformly continuous: (Using ε-δ-characterisation)

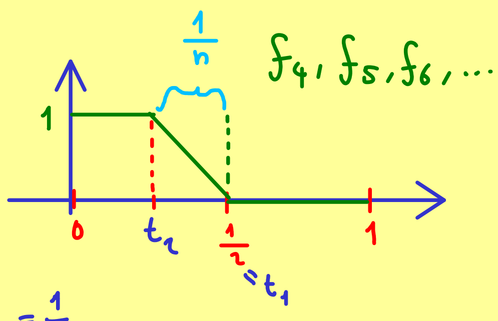
$$\underset{\varepsilon > 0}{\forall} \;\; \underset{\delta > 0}{\exists} \;\; \underset{t_1, t_2 \in [0,1]}{\forall} : \; |t_1 - t_2| < \delta \;\Rightarrow\; |f(t_1) - f(t_2)| < \varepsilon$$

$A \subseteq C([0,1])$ is called uniformly equicontinuous:

$$\underset{\varepsilon > 0}{\forall} \;\; \underset{\delta > 0}{\exists} \;\; \underset{t_1, t_2 \in [0,1]}{\forall} \;\; \underset{f \in A}{\forall} : \; |t_1 - t_2| < \delta \;\Rightarrow\; |f(t_1) - f(t_2)| < \varepsilon$$

or equivalently $\quad \sup_{f \in A} |f(t_1) - f(t_2)| \xrightarrow{|t_1 - t_2| \to 0} 0$

Examples: (a) $A := \{ f \in C([0,1]) \mid \|f\|_\infty \leq 1 \}$



$\sup_{f \in A} |f(t_1) - f(t_2)| \geq |f_n(t_1) - f_n(t_2)|$ for $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{2} - \frac{1}{n}$

$= 1$ (for $n \geq 4$)

$\Rightarrow A$ is not equicontinuous!

(b) $A := \{ f \in C([0,1]) \mid f \text{ continuously differentiable}, \; \|f'\|_\infty \leq 2 \}$

$$|f(t_1) - f(t_2)| \overset{\text{mean value theorem}}{\leq} |f'(\xi)| \cdot |t_1 - t_2| \leq 2 \cdot |t_1 - t_2|$$

$$\sup_{f \in A} |f(t_1) - f(t_2)| \leq 2 \cdot |t_1 - t_2| \xrightarrow{|t_1 - t_2| \to 0} 0 \;\Rightarrow\; A \text{ is uniformly equicontinuous}$$

Arzelà-Ascoli theorem: For $(C([0,1]), \|\cdot\|_\infty)$ holds: ($[0,1]$: could be any compact metric space)

$$A \subseteq C([0,1]) \text{ compact} \;\Longleftrightarrow\; A \text{ is } \left\{ \begin{array}{l} \text{closed +} \\ \text{bounded +} \\ \text{uniformly equicontinuous} \end{array} \right.$$