BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Functional analysis – part 20

Minkowski's inequality: Δ-inequality for $\|\cdot\|_p$ in $\ell^p(\mathbb{N})$:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \text{for all } x,y \in \ell^p(\mathbb{N}), \quad p \in [1,\infty)$$

Proof: For $p=1$: $\|x+y\|_1 = \sum_{j=1}^{\infty} \underbrace{|x_j+y_j|}_{\leq |x_j|+|y_j|} \leq \|x\|_1 + \|y\|_1$

For $p \in (1,\infty)$: Hölder conjugate $p' \in (1,\infty)$ $\quad \boxed{\frac{1}{p} + \frac{1}{p'} = 1}$

$$\frac{p}{p-1} = p'$$

$$\|x+y\|_p^p = \sum_{j=1}^{\infty} |x_j+y_j|^p = \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{|x_j+y_j|^p}_{\leq |x_j|+|y_j|} = (*)$$

$$(**) \quad \big(|x_j|+|y_j|\big)^p = \big(|x_j|+|y_j|\big)\big(|x_j|+|y_j|\big)^{p-1} = \underbrace{|x_j|}_{a_j}\underbrace{\big(|x_j|+|y_j|\big)^{p-1}}_{b_j} + \underbrace{|y_j|}_{c_j}\big(|x_j|+|y_j|\big)^{p-1}$$

$\leadsto a,b,c \in \mathbb{F}^n$

Hölder: $\|ab\|_1 \leq \|a\|_p \cdot \underbrace{\|b\|_{p'}}_{= \left(\sum_{j=1}^{n} \left|(|x_j|+|y_j|)^{p-1}\right|^{p'}\right)^{\frac{1}{p'}} = \left(\sum_{j=1}^{n} (|x_j|+|y_j|)^p\right)^{\frac{1}{p'}}}$

$$(***) \quad \sum_{j=1}^{n} \big(|x_j|+|y_j|\big)^p \leq \|a\|_p \cdot \|b\|_{p'} + \|c\|_p \cdot \|b\|_{p'} = \big(\|a\|_p + \|c\|_p\big) \cdot \left(\sum_{j=1}^{n} \big(|x_j|+|y_j|\big)^p\right)^{\frac{1}{p'}}$$

$$\Rightarrow \left(\sum_{j=1}^{n} \big(|x_j|+|y_j|\big)^p\right)^{\underbrace{1-\frac{1}{p'}}_{\frac{1}{p}}} \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

$$\xrightarrow[+ (*)]{n\to\infty} \|x+y\|_p \leq \|x\|_p + \|y\|_p$$