The Bright Side of
Mathematics

Functional analysis – part 27

Bounded inverse theorem: X, Y Banach spaces, $T \in \mathcal{B}(X,Y)$.

Then: T bijective $\Rightarrow$ $T^{-1} \in \mathcal{B}(Y,X)$ (It's continuous)

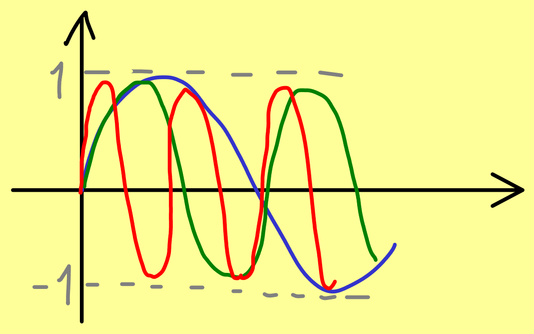
Proof: T bijective $\underset{\text{open mapping theorem}}{\Rightarrow}$ T open map $\Rightarrow$ T^{-1} continuous $\square$

Counterexample: $X = \left(C([0,1]), \|\cdot\|_\infty\right)$, $Y = \left(\{ f \in C^1([0,1]) \mid f(0) = 0 \}, \|\cdot\|_\infty\right) \rightarrow$ not complete

$$(Tf)(t) = \int_0^t f(s)\, ds \quad \text{linear and bounded and bijective}$$

$$\|Tf\|_\infty = \sup_{t \in [0,1]} \left| \int_0^t f(s)\, ds \right| \leq \|f\|_\infty \Rightarrow \|T\|_{X \to Y} \leq 1$$

Take $f_k(t) = \sin(kt)$



$$(Tf_k)(t) = \underbrace{\frac{1}{k}\left(1 - \cos(kt)\right)}_{g_k(t)}$$

$$T^{-1} g_k = f_k \Rightarrow \|T^{-1}\|_{Y \to X} \geq \frac{\|T^{-1} g_k\|_\infty}{\|g_k\|_\infty} = \frac{\overbrace{\|f_k\|_\infty}^{=1}}{\underbrace{\|g_k\|_\infty}_{\leq \frac{2}{k}}} \geq \frac{k}{2} \xrightarrow{k \to \infty} \infty$$

$\Rightarrow$ T^{-1} not continuous