Functional analysis – part 30

X complex Banach space

$T: X \to X$ bounded linear operator

$$\sigma(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda) \text{ not invertible} \}$$

$$\rho(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda) \text{ invertible} \}$$

Proposition: (a) $\rho(T)$ is an open set

$\sigma(T)$ is a closed set

(b) For $\lambda \in \rho(T)$: $\quad \left\| (T-\lambda)^{-1} \right\| \geq \dfrac{1}{\operatorname{dist}(\lambda, \sigma(T))}$

(c) The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$\lambda \longmapsto (T-\lambda)^{-1}$ is analytical.

Locally, it can be expressed as a Taylor series.

Proof: Choose $\lambda_0 \in \rho(T)$ and set $C := \left\| (T-\lambda_0)^{-1} \right\|$, $\varepsilon := \dfrac{1}{C}$

Let's take any $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon$.

Calculate: $T - \lambda = (T - \lambda_0) - (\lambda - \lambda_0) = (T-\lambda_0)\Big(I - \underbrace{(\lambda - \lambda_0)\cdot (T-\lambda_0)^{-1}}_{S} \Big)$

$\|S\| < \varepsilon \cdot C = 1$

> Neumann series: $(I - S)$ with $\|S\| < 1$ is invertible because
>
> $(I - S) \cdot \sum_{k=0}^{n} S^k = (I - S^{n+1}) \xrightarrow{n \to \infty} I \quad \Rightarrow (I-S)^{-1} = \sum_{k=0}^{\infty} S^k$

$\Rightarrow \quad T - \lambda$ is invertible $\Rightarrow \lambda \in \rho(T) \Rightarrow \rho(T)$ is open (a) ✓

Also: $(T-\lambda)^{-1} = (I - S)^{-1} (T-\lambda_0)^{-1} = \sum_{k=0}^{\infty} S^k \cdot (T-\lambda_0)^{-1}$

$$= \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k \cdot (T-\lambda_0)^{-k} \cdot (T-\lambda_0)^{-1} = \sum_{k=0}^{\infty} (T-\lambda_0)^{-(k+1)} \cdot (\lambda - \lambda_0)^k$$

(c) ✓

Now for $\lambda \in \sigma(T) \overset{\text{above}}{\Longrightarrow} |\lambda - \lambda_0| \geq \varepsilon \Rightarrow \dfrac{1}{|\lambda - \lambda_0|} \leq C = \left\| (T-\lambda_0)^{-1} \right\|$

$$\frac{1}{\operatorname{dist}(\lambda_0, \sigma(T))} = \frac{1}{\inf\limits_{\lambda \in \sigma(T)} |\lambda - \lambda_0|} = \sup_{\lambda \in \sigma(T)} \frac{1}{|\lambda - \lambda_0|} \leq \left\| (T-\lambda_0)^{-1} \right\|$$

(b) ✓