The Bright Side of Mathematics

The following pages cover the whole Functional Analysis course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Functional analysis - part 1

Linear algebra
dim = ∞

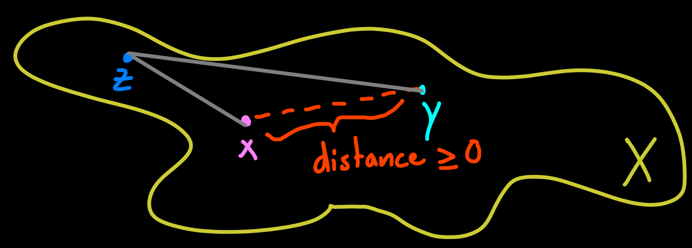
+

Real and complex analysis

Functional analysis
(function spaces , sequences , ...)
= Study of topological - algebraic structures

Metric spaces

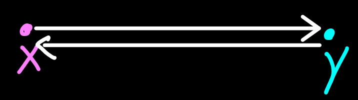
X set



a metric : $d : X \times X \longrightarrow [0, \infty)$

(1) $d(x,y) = 0 \iff x = y$

(2) $d(x,y) = d(y,x)$

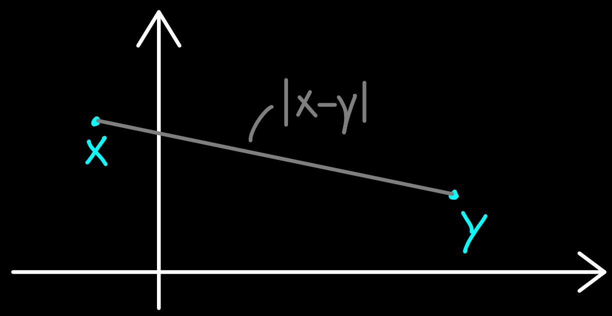


(3) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

Functional analysis – part 2

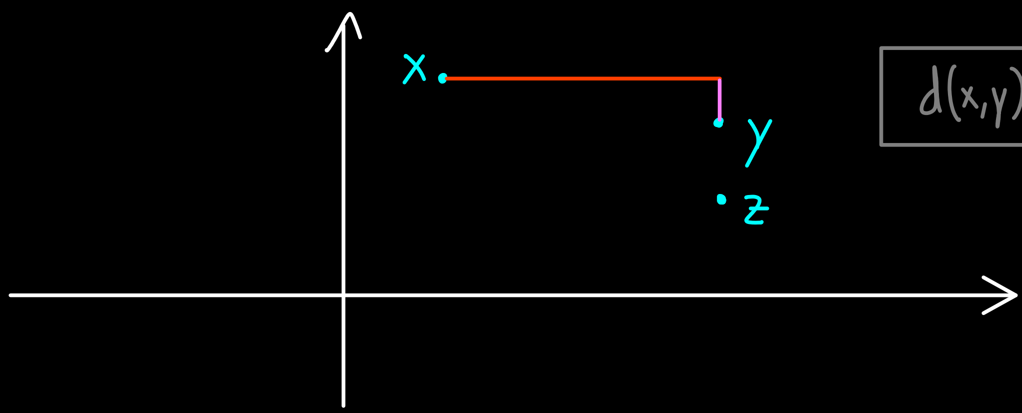
X set $+$ $d : X \times X \longrightarrow [0, \infty)$ metric $=$ metric space (X, d)

Examples: (a) $X = \mathbb{C}$, $d(x,y) = |x-y|$



(b) $X = \mathbb{R}^n$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$ (Euclidean metric)

(c) $X = \mathbb{R}^n$, $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_n-y_n|\}$



$\boxed{d(x,y) = d(x,z)}$

(d) X any set $(\neq \emptyset)$, $d(x,y) = \begin{cases} 0 , & x = y \\ 1 , & x \neq y \end{cases}$ $\boxed{\text{discrete metric}}$

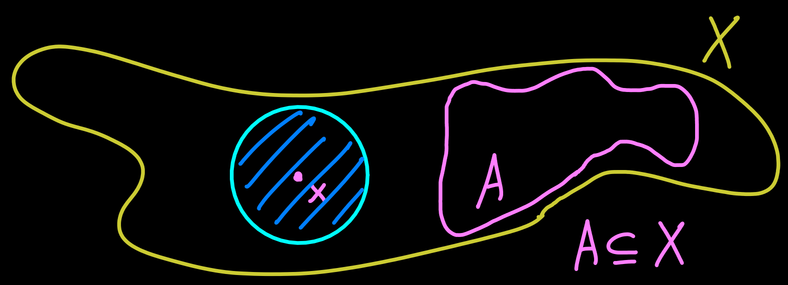
d is a metric: (1) ✓, (2) ✓ , (3) Δ-inequality: $x, y, z \in X$

First case: $x = y$: $d(x,y) = 0 \leq d(x,z) + d(z,y)$ ✓

Second case: $x \neq y$: $d(x,y) = 1 = \left\{ \begin{matrix} d(x,z) \\ \text{or} \\ d(z,y) \end{matrix} \right\} \leq d(x,z) + d(z,y)$ ✓

Functional analysis - part 3

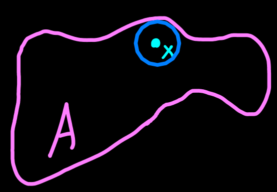
(X, d) metric space



$$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$$ (open ball of radius $\varepsilon > 0$ centered at x)

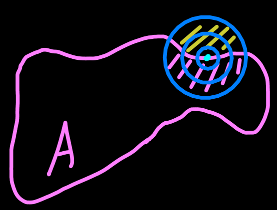
Notions:

(1) Open sets:



$A \subseteq X$ is called <u>open</u> if for each $x \in A$ there is an open ball with $B_\varepsilon(x) \subseteq A$.

(2) Boundary points:

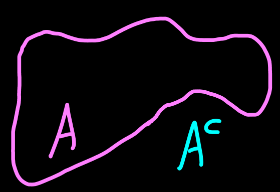


$A \subseteq X$. $x \in X$ is called a <u>boundary point for A</u> if for all $\varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$ $\quad [A^c := X \setminus A]$

$$\partial A := \{ x \in X \mid x \text{ is boundary point for } A \}$$

Remember: A open $\Longleftrightarrow$ $A \cap \partial A = \emptyset$

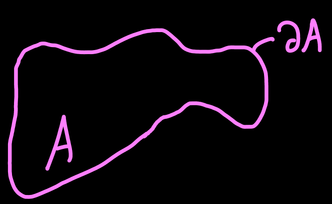
(3) Closed sets:



$A \subseteq X$ is called <u>closed</u> if $A^c := X \setminus A$ is open.

Remember: A closed $\Longleftrightarrow$ $A \cup \partial A = A$

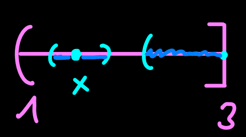
(4) Closure:



$\overline{A} := A \cup \partial A$ (always closed!)

Example: $X := (1, 3] \cup (4, \infty)$, $d(x,y) := |x-y|$, (X, d) is a metric space

(a) $A := (1,3] \subseteq X$ open?



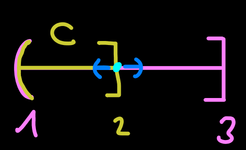
For $x \in A$, $x \neq 3$, define $\varepsilon := \frac{1}{2}\min(|1-x|, |3-x|)$. Then $B_\varepsilon(x) \subseteq A$.

For $x = 3$: $B_1(x) = \{ y \in X \mid d(x,y) < 1 \} = (2,3] \subseteq A$

$\Rightarrow$ A is open

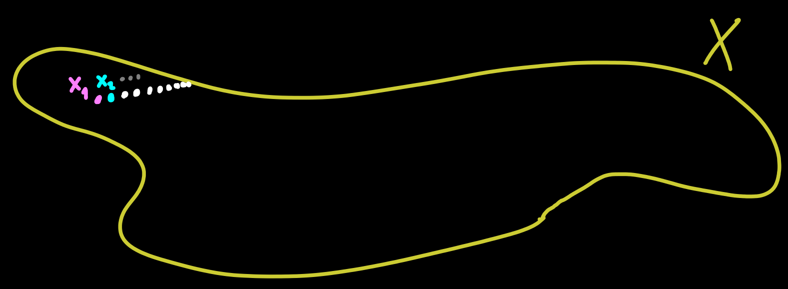
(b) A is also closed!

(c) $C := (1,2]$, $\partial C = \{2\}$, $\overline{C} = C$



Functional analysis - part 4

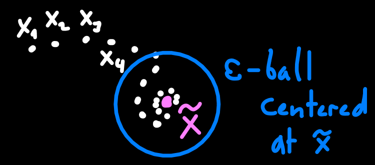
(X, d) metric space



Sequence in X: $(x_1, x_2, x_3, \dots)$ or $(x_n)_{n \in \mathbb{N}}$ or $x : \mathbb{N} \longrightarrow X$, $n \longmapsto x_n$ map

Convergence: A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is called convergent if there is $\tilde{x} \in X$ with

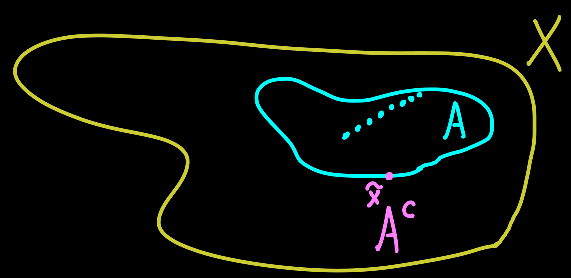
$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad d(x_n, \tilde{x}) < \varepsilon .$$



We write: $\quad x_n \xrightarrow{n \to \infty} \tilde{x} \quad$ or $\quad \lim_{n \to \infty} x_n = \tilde{x}$.

Proposition: $A \subseteq X$ is closed

$\Longleftrightarrow$ For every convergent sequence $(a_n)_{n \in \mathbb{N}} \subseteq A$, one has $\lim_{n \to \infty} a_n \in A$



Proof: $(\Leftarrow)$: Show it by contraposition! Assume A is not closed.

$\Rightarrow$ $A^c := X \setminus A$ is not open.

$\Rightarrow$ There is an $\tilde{x} \in A^c$ with $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

$\Rightarrow$ There is a sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in B_{\frac{1}{n}}(\tilde{x}) \cap A$

$\Rightarrow$ $\lim_{n \to \infty} a_n = \tilde{x} \notin A$

$(\Rightarrow)$: Show it by contraposition! Assume there is $(a_n)_{n \in \mathbb{N}} \subseteq A$ with $\tilde{x} := \lim_{n \to \infty} a_n \notin A$.

$\Rightarrow$ $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\varepsilon > 0$. $\Rightarrow$ A^c is not open $\Rightarrow$ A is not closed

Functional analysis - part 5

Example: $X = (0,3)$ with $d(x,y) = |x-y|$

$(0,3)$ is closed:

- complement $\emptyset$ is open
- each convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq (0,3)$ (with limit $\tilde{x} \in X$) satisfies $\tilde{x} \in (0,3)$

What is about the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$?

- sequence in X
- $d(x_n, x_m) \xrightarrow{n,m\to\infty} 0$
- it does not converge $\Rightarrow (X,d)$ is not complete

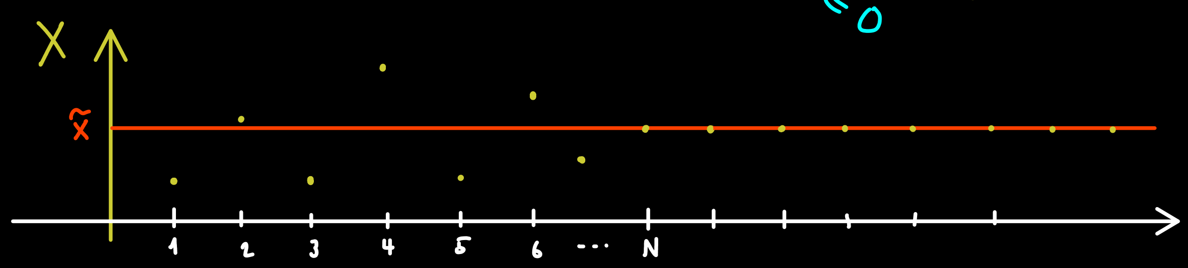
Definition: Let (X,d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called Cauchy sequence if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n,m \geq N : d(x_n, x_m) < \varepsilon$.

(X,d) is called complete if all Cauchy sequences converge.

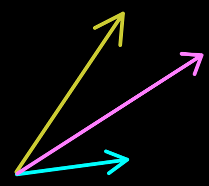
Example: (a) $X = [0,3]$ with $d(x,y) = |x-y|$ is complete.

(b) $X = (0,3)$ with $d(x,y) = \begin{cases} 1 & , x \neq y \\ 0 & , x = y \end{cases}$ is complete.

Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Take $\varepsilon = \frac{1}{2}$. Then there is an $N \in \mathbb{N}$ such that for all $n,m \geq N$, we have $\underbrace{d(x_n, x_m)}_{=0} < \frac{1}{2}$. Hence $x_n = x_m$.



Functional analysis - part 6



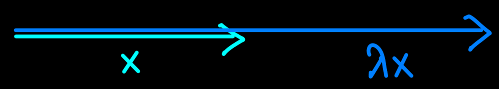
Definition: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let X be a $\mathbb{F}$-vector space.

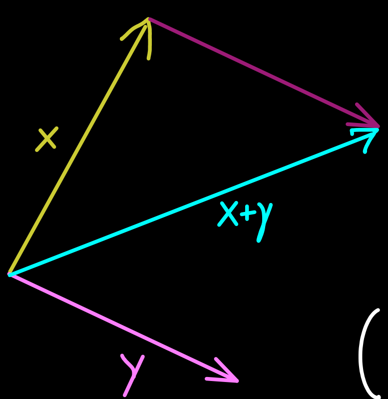
A map $\|\cdot\| : X \longrightarrow [0, \infty)$ is called <u>norm</u> if

(a) $\|x\| = 0 \iff x = 0$ (positive definite)

(b) $\|\lambda \cdot x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{F}$, $x \in X$ (absolutely homogeneous)

absolute value in $\mathbb{R}$ or $\mathbb{C}$





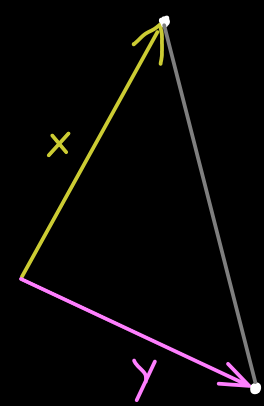
(c) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality)

$(X, \|\cdot\|)$ is then called a <u>normed space</u>.

Important: If $\|\cdot\|$ is a norm for the $\mathbb{F}$-vector space X, then

$$d_{\|\cdot\|}(x,y) := \|x - y\|$$ defines

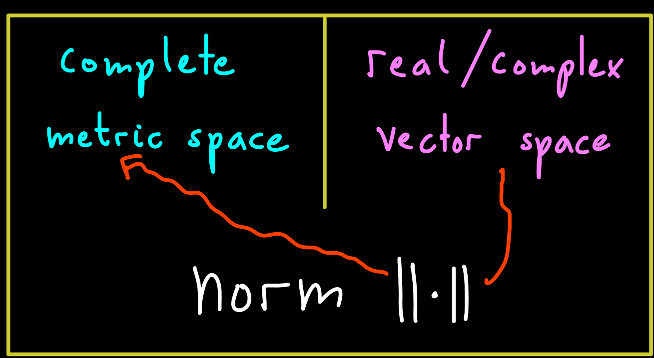
a metric for the set X.



If $(X, d_{\|\cdot\|})$ is a <u>complete</u> metric space,

then the normed space $(X, \|\cdot\|)$ is called a <u>Banach space</u>.

Banach space:



Functional analysis - part 7

Banach space:

complete metric space	real/complex vector space
norm $\|\cdot\|$	

<u>Examples:</u>

(1) $\mathbb{R}$ is a one-dimensional real vector space

$|\cdot| : \mathbb{R} \longrightarrow [0,\infty)$ is a norm. $d_{|\cdot|}(x,y) := |x-y|$ is a metric.

$\Longrightarrow (\mathbb{R}, |\cdot|)$ is a Banach space

(2) $X = \{0\}$, zero-dimensional real vector space.

$\|\cdot\| : X \longrightarrow [0,\infty)$ defined by $\|0\| := 0$.

$\Rightarrow (X, \|\cdot\|)$ is a Banach space.

(3) Let $\underset{\ell^p}{\overset{\shortparallel}{\ell^p(\mathbb{N},\mathbb{F})}}$ (where $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, $p \in [1,\infty)$)

be defined as all sequences $x = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \qquad \text{(converges!)}$$

Then $\|\cdot\|_p : \ell^p \longrightarrow [0,\infty)$ with $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$ is a norm!

(Show later!)

<u>Claim:</u> $(\ell^p, \|\cdot\|_p)$ is a Banach space

<u>Proof:</u> • ℓ^p is an $\mathbb{F}$-vector space and $\|\cdot\|_p$ is a norm on it (see later).

• Completeness: Let $(x^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in ℓ^p.

$$
\begin{array}{lcl}
x^{(1)} & = & (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \boxed{x_4^{(1)}}, x_5^{(1)}, x_6^{(1)}, x_7^{(1)}, x_8^{(1)} \dots) \\
x^{(2)} & = & (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \boxed{x_4^{(2)}}, x_5^{(2)}, x_6^{(2)}, x_7^{(2)}, x_8^{(2)} \dots) \\
x^{(3)} & = & (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \boxed{x_4^{(3)}}, x_5^{(3)}, x_6^{(3)}, x_7^{(3)}, x_8^{(3)} \dots) \\
\vdots\ ? & & \quad\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
\tilde{x} & := & (\tilde{x}_1\ \ \tilde{x}_2\ \ \tilde{x}_3\ \ \tilde{x}_4\ \ \tilde{x}_5\ \ \tilde{x}_6\ \ \tilde{x}_7\ \ \tilde{x}_8 \dots)
\end{array}
$$

$$|x_m^{(k)} - x_m^{(\ell)}|^p \le \sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(\ell)}|^p = \| x^{(k)} - x^{(\ell)} \|_p^p$$

Cauchy sequence: $\forall \varepsilon > 0 \ \ \exists K \in \mathbb{N} \ \ \forall k, \ell \ge K: \ \| x^{(k)} - x^{(\ell)} \|_p < \varepsilon$

$|x_m^{(k)} - x_m^{(\ell)}| \le$ $\quad \Rightarrow \ (x_m^{(k)})_{k\in\mathbb{N}}$ Cauchy seq. in $\mathbb{F}$

$\Longrightarrow \ (x_m^{(k)})_{k\in\mathbb{N}}$ has a limit $\tilde{x}_m \in \mathbb{F}$

Let $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that $\forall k, \ell \ge K: \ \| x^{(k)} - x^{(\ell)} \|_p < \varepsilon' =: \frac{\varepsilon}{2}$

$$\| x^{(k)} - \tilde{x} \|_p^p = \sum_{n=1}^{\infty} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N\to\infty} \sum_{n=1}^{N} |x_n^{(k)} - \tilde{x}_n|^p = \lim_{N\to\infty} \lim_{\ell\to\infty} \underbrace{\sum_{n=1}^{N} |x_n^{(k)} - x_n^{(\ell)}|^p}_{<(\varepsilon')^p}$$

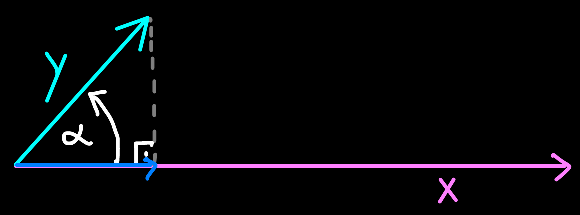
Then for all $k \ge K$: $\quad \| x^{(k)} - \tilde{x} \|_p \le (\varepsilon') < \varepsilon$

And $\tilde{x} = \underbrace{\tilde{x} - x^{(k)}}_{\in \ell^p} + \underbrace{x^{(k)}}_{\in \ell^p} \in \ell^p$ (it's a vector space!)

Functional analysis - part 8

- metric $\longrightarrow$ measures distances
- norm $\longrightarrow$ measures distances, lengths
- inner product $\longrightarrow$ measures distances, lengths, angles

"$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos(\alpha)$"



Definition: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let X be an $\mathbb{F}$-vector space.

A map $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{F}$ is called an <u>inner product</u> on X if

(1) $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0 \iff x = 0$ [positive definite]

(2) $\langle x, y \rangle = \langle y, x \rangle$ for $\mathbb{F} = \mathbb{R}$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$ for $\mathbb{F} = \mathbb{C}$

for all $x, y \in X$ [(conjugate) symmetric]

(3) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$ for all $x, y_1, y_2 \in X$

$\langle x, \lambda \cdot y \rangle = \lambda \cdot \langle x, y \rangle$ for all $x, y \in X$, $\lambda \in \mathbb{F}$ [linear in the 2nd argument]

If $\langle \cdot, \cdot \rangle$ is an inner product, then $\|x\|_{\langle \cdot,\cdot \rangle} := \sqrt{\langle x, x \rangle}$ defines norm.

Definition: $(X, \langle \cdot, \cdot \rangle)$ is called a <u>Hilbert space</u> if $(X, \|\cdot\|_{\langle \cdot,\cdot \rangle})$ is a Banach space.

Functional analysis - part 9

Examples of Hilbert spaces

(a) $\mathbb{R}^n$, $\mathbb{C}^n$ with $\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$

(b) $\ell^2(\mathbb{N}, \mathbb{F})$ with $\langle x, y\rangle = \sum_{i=1}^{\infty} \overline{x_i}\, y_i$

inner product

Not a Hilbert space $\longrightarrow$ (c) $C([0,1], \mathbb{F})$ with $\langle f, g\rangle = \int_0^1 \overline{f(t)} g(t)\, dt$

$\left(\ell^2(\mathbb{N}, \mathbb{F}), \langle\cdot,\cdot\rangle\right)$ is a Hilbert space: $\quad \langle\cdot,\cdot\rangle : \ell^2 \times \ell^2 \longrightarrow \mathbb{F}$ later!

(1) positive definite: $\langle x, x\rangle = \sum_{i=1}^{\infty} \overline{x_i}\, x_i = \sum_{i=1}^{\infty} |x_i|^2 \geq 0$

and $\langle x, x\rangle = 0 \Rightarrow |x_i|^2 = 0$ for all $i \in \mathbb{N}$

$\Rightarrow x_i = 0$ for all $i \in \mathbb{N} \Rightarrow x = 0$.

(2) (conjugate) symmetric: $\overline{\langle y, x\rangle} = \sum_{i=1}^{\infty} \overline{\overline{y_i}\, x_i} = \sum_{i=1}^{\infty} y_i\, \overline{x_i} = \langle x, y\rangle$

(3) linear in the 2^{nd} argument: $\langle x, y+z\rangle = \sum_{i=1}^{\infty} \overline{x_i}(y_i + z_i) = \sum_{i=1}^{\infty} \overline{x_i}\, y_i + \sum_{i=1}^{\infty} \overline{x_i}\, z_i$

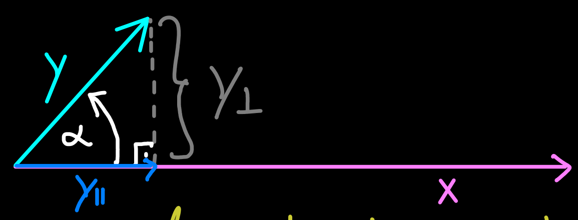
$= \langle x, y\rangle + \langle x, z\rangle$

$\langle x, \lambda \cdot y\rangle = \sum_{i=1}^{\infty} \overline{x_i}(\lambda y_i) = \lambda \cdot \sum_{i=1}^{\infty} \overline{x_i}\, y_i = \lambda \cdot \langle x, y\rangle$

Functional analysis - part 10

Cauchy-Schwarz inequality: Let $(X, \langle \cdot,\cdot \rangle)$ be an inner product space and $\|x\| := \sqrt{\langle x, x\rangle}$. Then for all $x, y \in X$:

$$|\langle x,y\rangle| \leq \|x\| \cdot \|y\|$$



and $$|\langle x,y\rangle| = \|x\|\cdot\|y\| \iff x, y \text{ linearly dependent}$$

Proof: 1st case: $x = 0$: $\quad |\langle x,y\rangle| = 0 = \|x\|\cdot\|y\|$ ✓

2nd case $x \neq 0$: $\quad \hat{x} := \frac{x}{\|x\|}$, $\quad y_\parallel := \langle \hat{x}, y\rangle \hat{x}$, $\quad y_\perp := y - y_\parallel$

$$0 \leq \|y_\perp\|^2 = \|y - y_\parallel\|^2 = \|y - \langle \hat{x},y\rangle\hat{x}\|^2 = \langle y - \langle \hat{x},y\rangle\hat{x}, y - \langle \hat{x},y\rangle \hat{x}\rangle$$

$$= \langle y - \langle\hat{x},y\rangle\hat{x}, y\rangle - \langle y - \langle \hat{x},y\rangle\hat{x}, \langle\hat{x},y\rangle\hat{x}\rangle$$

$$= \langle y,y\rangle - \langle\langle\hat{x},y\rangle\hat{x}, y\rangle - \langle y, \langle\hat{x},y\rangle\hat{x}\rangle + \langle\langle\hat{x},y\rangle\hat{x}, \langle\hat{x},y\rangle\hat{x}\rangle$$

$$= \|y\|^2 - \left(\langle\langle\hat{x},y\rangle\hat{x}, y\rangle + \overline{\langle\langle\hat{x},y\rangle\hat{x}, y\rangle}\right) + \|\langle\hat{x},y\rangle\hat{x}\|^2$$

$$= \|y\|^2 - \left(2 \cdot \mathrm{Re}\big(\underbrace{\langle\langle\hat{x},y\rangle\hat{x}, y\rangle}_{\overline{\langle\hat{x},y\rangle}\langle\hat{x},y\rangle = |\langle\hat{x},y\rangle|^2}\big)\right) + |\langle\hat{x},y\rangle|^2 \cdot \underbrace{\|\hat{x}\|^2}_{=1}$$

$$= \|y\|^2 - 2|\langle\hat{x},y\rangle|^2 + |\langle\hat{x},y\rangle|^2 = \|y\|^2 - |\langle\hat{x},y\rangle|^2$$

$$\Rightarrow \quad \|y\|^2 \geq |\langle\hat{x},y\rangle|^2 = \left|\left\langle \tfrac{x}{\|x\|}, y\right\rangle\right|^2 = \frac{1}{\|x\|^2}|\langle x,y\rangle|^2$$

$$\Rightarrow \quad \|x\|\cdot\|y\| \geq |\langle x,y\rangle|$$

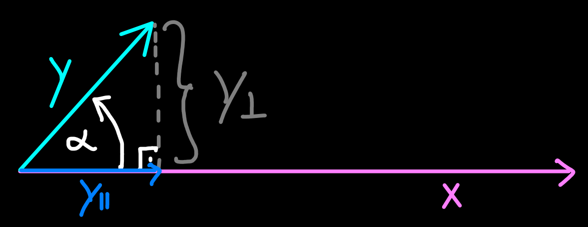
Δ-inequality for $\|\cdot\|$:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

$$\overset{\text{Cauchy Schwarz}}{\leq} \|x\|^2 + 2\cdot\|x\|\cdot\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Functional analysis - part 11



Orthogonality: Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space.

(a) $x, y \in X$ are called <u>orthogonal</u> if $\langle x, y\rangle = 0$. Write $x \perp y$.

(b) For $U, V \subseteq X$, write $U \perp V$ if $x \perp y$ for all $x \in U, y \in V$.

(c) For $U \subseteq X$, the <u>orthogonal complement of U</u> is

$$\underbrace{U^\perp}_{\text{is always a subspace in } X} := \{ x \in X \mid \langle x, u\rangle = 0 \text{ for all } u \in U \}$$

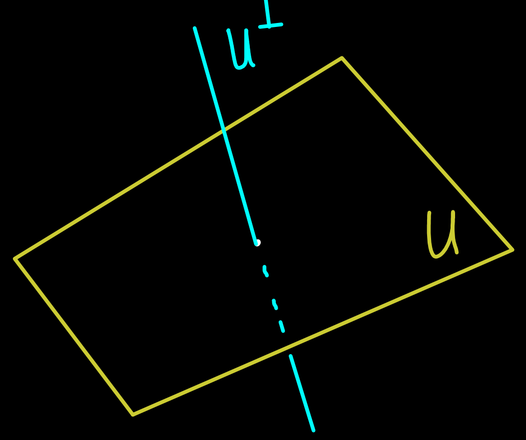
Remark: (1) $\{0\}^\perp = X$, $X^\perp = \{0\}$

(2) $U \subseteq V \quad \Rightarrow \quad U^\perp \supseteq V^\perp$

Proof: $x \in V^\perp \Rightarrow \langle x, v\rangle = 0$ for all $v \in V$

$\overset{U \subseteq V}{\Longrightarrow} \langle x, u\rangle = 0$ for all $u \in U \quad \Rightarrow x \in U^\perp$

(3) If $x \perp y$, then $\|x+y\|^2_{\langle\cdot,\cdot\rangle} = \|x\|^2_{\langle\cdot,\cdot\rangle} + \|y\|^2_{\langle\cdot,\cdot\rangle}$ (Pythagorean theorem)



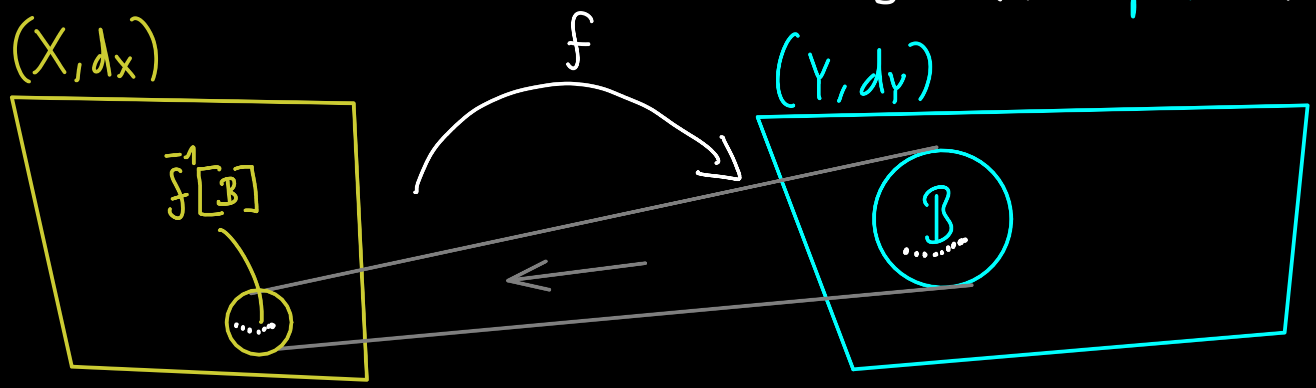
$U^\perp$ is always closed

Functional analysis – part 12

Continuity for metric spaces: (X, d_X), (Y, d_Y) two metric spaces.

A map $f: X \longrightarrow Y$ is called:

- continuous if $f^{-1}[B]$ is open (in X) for all open sets $B \subseteq Y$.



- sequentially continuous if for all $\tilde{x} \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} \tilde{x}$ holds $f(x_n) \xrightarrow{n\to\infty} f(\tilde{x})$.

Fact: For metric spaces, continuous and sequentially continuous are equivalent.

Examples: (a) (X, d_X) discrete metric space, (Y, d_Y) any metric space

$\Rightarrow$ all $f: X \to Y$ are continuous

(b) (X, d_X), (Y, d_Y) metric spaces, $y_0 \in Y$ fixed.

$\Rightarrow$ $f: X \to Y$, $x \mapsto y_0$ is always continuous.

(c) $(X, \|\cdot\|)$ normed space, $Y = \mathbb{R}$ with standard metric

$$\Rightarrow \quad f: X \to \mathbb{R}, \; x \mapsto \|x\| \quad \text{is continuous}$$

Proof: Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$f(x_n) = \|x_n\| = \|x_n - \tilde{x} + \tilde{x}\| \overset{\Delta\text{-inequ.}}{\leq} \|x_n - \tilde{x}\| + \|\tilde{x}\| = \underbrace{d(x_n, \tilde{x})}_{\xrightarrow{n\to\infty} 0} + f(\tilde{x})$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) \leq f(\tilde{x})$$

$$f(\tilde{x}) = \|\tilde{x}\| = \|\tilde{x} - x_n + x_n\| \overset{\Delta\text{-inequ.}}{\leq} \|\tilde{x} - x_n\| + \|x_n\| = d(\tilde{x}, x_n) + f(x_n)$$

$$\Rightarrow f(\tilde{x}) \leq \lim_{n\to\infty} f(x_n) \qquad \square$$

(d) $(X, \langle\cdot,\cdot\rangle)$ inner product space, $Y = \mathbb{C}$ with the standard metric, $x_0 \in X$ fixed.

$$\Rightarrow \quad f: X \to \mathbb{C}, \; x \mapsto \langle x_0, x\rangle \quad \text{is continuous}$$

Proof: Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$|f(x_n) - f(\tilde{x})| = |\langle x_0, x_n\rangle - \langle x_0, \tilde{x}\rangle| = |\langle x_0, x_n - \tilde{x}\rangle|$$

$$\overset{\text{C.S.}}{\leq} \|x_0\| \cdot \underbrace{\|x_n - \tilde{x}\|}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to\infty} 0$$

Analogously, $g: X \to \mathbb{C}$, $x \mapsto \langle x, x_0\rangle$ is continuous.

Claim: $(X, \langle\cdot,\cdot\rangle)$ inner product space, $U \subseteq X$. Then $U^\perp$ is closed.

Proof: Let $(x_n)_{n\in\mathbb{N}} \subseteq U^\perp$ with limit $\tilde{x} \in X$.

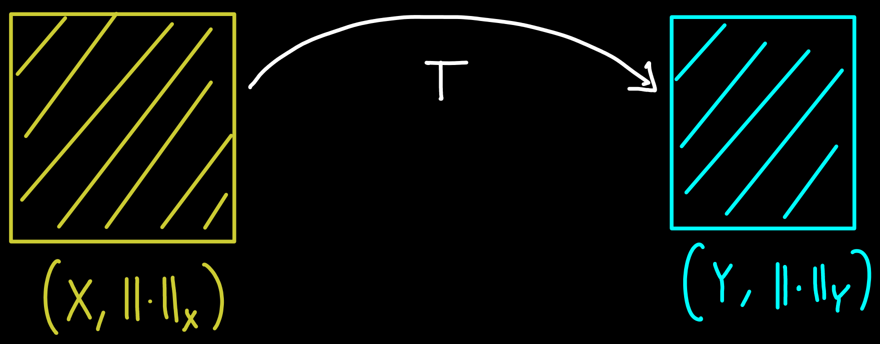
$$\Rightarrow \langle x_n, u\rangle = 0 \quad \text{for all } u \in U$$

$$\Rightarrow \lim_{n\to\infty} \langle x_n, u\rangle = 0 \quad \text{for all } u \in U$$

$$\Rightarrow \langle \tilde{x}, u\rangle = 0 \quad \text{for all } u \in U \quad \Rightarrow \tilde{x} \in U^\perp \qquad \square$$

Functional analysis – part 13

Operator:



$T: X \longrightarrow Y$:

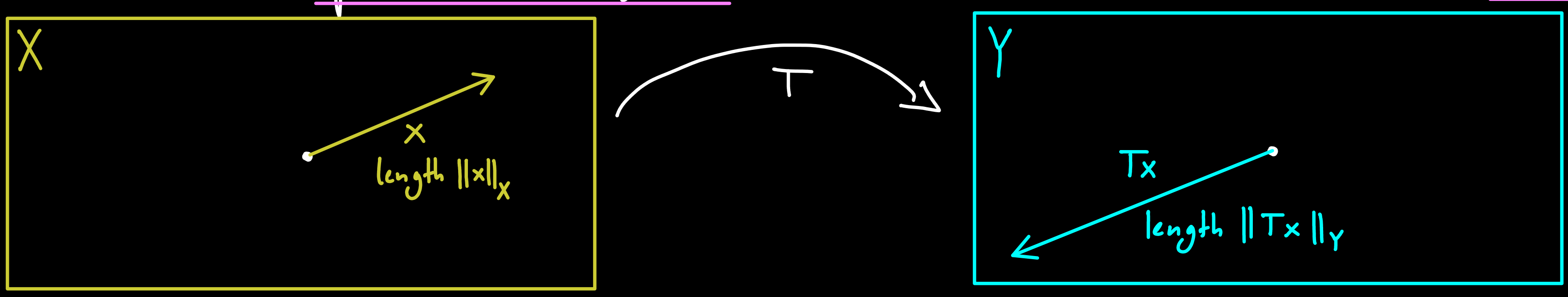
- linear (conserves the algebraic structure)
- continuous (bounded) (conserves the topological structure)

<u>Definition:</u> $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ two normed spaces, $T: X \longrightarrow Y$ linear

$\begin{cases} T(x+\tilde{x}) = Tx + T\tilde{x} \\ T(\lambda x) = \lambda Tx \end{cases}$ for all $x, \tilde{x} \in X$, $\lambda \in \mathbb{F}$

$$\|T\| = \|T\|_{X \to Y} := \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X,\ x \neq 0 \right\}$$

is called the <u>operator norm of T</u>. If $\|T\| < \infty$, T is called <u>bounded</u>.



Proposition: Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ two normed spaces, $T: X \longrightarrow Y$ linear.
Then the following claims are equivalent:

(a) T is continuous.
(b) T is continuous at $x = 0$.
(c) T is bounded.

Proof: (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c) : $(*)$ For all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} 0$, we have $Tx_n \xrightarrow{n\to\infty} 0$.

Claim: $(*) \Rightarrow$ [There is a $\delta > 0$ such that $\|Tx\|_Y < 1$ for all $x \in X$ with $\|x\|_X < \delta$] $(\divideontimes)$

Proof of the claim: $\neg(\divideontimes) \Rightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in X$ with $\|x_n\|_X < \frac{1}{n}$ and $\|Tx_n\|_Y \geq 1$ $\Rightarrow \neg(*)$

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|Tx\|_Y \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}}{\|x\|_X \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}} = \frac{\|T(\frac{\delta}{2}\frac{x}{\|x\|_X})\|_Y}{\underbrace{\|\frac{\delta}{2}\frac{x}{\|x\|_X}\|_X}_{=\frac{\delta}{2}}} \leq \frac{2}{\delta}$$

$$\Rightarrow \|T\| = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X,\ x \neq 0\right\} \leq \frac{2}{\delta} < \infty$$

(c) $\Rightarrow$ (a) : Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be convergent with limit $\tilde{x} \in X$. Then

$$\|Tx_n - T\tilde{x}\|_Y = \|T(x_n - \tilde{x})\|_Y \leq \|T\| \cdot \|x_n - \tilde{x}\|_X \xrightarrow{n\to\infty} 0 \qquad \square$$

Functional analysis – part 14

Example: $X = (C([0,1], \mathbb{F}), \|\cdot\|_\infty)$, $Y = (\mathbb{F}, |\cdot|)$

For $g \in X$ with $g(t) \neq 0$ for all $t \in [0,1]$, define

$$T_g : X \longrightarrow Y \quad \text{by} \quad T_g(f) := \int_0^1 g(t) \cdot f(t)\, dt$$

What is $\|T_g\|$?

$$\|T_g\| = \sup \left\{ \frac{|T_g(f)|}{\|f\|_\infty} \;\middle|\; f \in X,\ f \neq 0 \right\}$$

$$= \sup \left\{ |T_g(f)| \;\middle|\; f \in X,\ \|f\|_\infty = 1 \right\}$$

$$= \sup \left\{ \underbrace{\left| \int_0^1 g(t) \cdot f(t)\, dt \right|}_{\leq \int_0^1 |g(t)| \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty = 1} dt} \;\middle|\; f \in X,\ \|f\|_\infty = 1 \right\}$$

$$\leq \int_0^1 |g(t)|\, dt < \infty$$

Check the other inequality: $h(t) := \dfrac{\overline{g(t)}}{|g(t)|}$ with $\|h\|_\infty = 1$

$$\|T_g\| \geq |T_g(h)| = \left| \int_0^1 g(t) \frac{\overline{g(t)}}{|g(t)|}\, dt \right| = \int_0^1 \frac{|g(t)|^{\cancel{2}}}{\cancel{|g(t)|}}\, dt = \int_0^1 |g(t)|\, dt$$

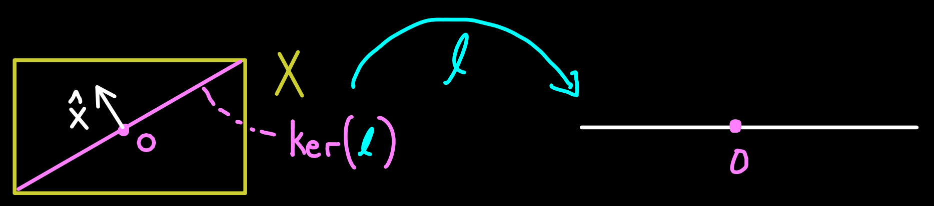
Functional analysis – part 15

Riesz representation theorem

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then for each continuous linear map $\ell : X \to \mathbb{F}$ (a continuous linear functional) there is exactly one $x_\ell \in X$ such that $\ell(x) = \langle x_\ell, x \rangle$ for all $x \in X$ and $\|\ell\|_{X \to \mathbb{F}} = \|x_\ell\|_X$.

[In physics $\ell = \langle \psi |$]

Proof: (1) Existence:



First case: $\ker(\ell) = X \Rightarrow x_\ell = 0$

Second case: $\ker(\ell) \neq X \rightsquigarrow x_\ell \in \underbrace{\ker(\ell)^\perp}_{\supsetneq \{0\}}$ ← true because $\ker(\ell)$ is closed and "orthogonal projections" exist in Hilbert spaces → later

Kernel is preimage of closed set $\{0\}$ $\overset{\text{continuity}}{\rightsquigarrow}$ Kernel is closed.

Choose $\hat{x} \in \ker(\ell)^\perp$ with $\|\hat{x}\|_X = 1$. Set $x_\ell := \overline{\ell(\hat{x})} \cdot \hat{x}$

$$\ell(x) = \ell\left(x - \frac{\ell(x)}{\ell(\hat{x})}\hat{x} + \frac{\ell(x)}{\ell(\hat{x})}\hat{x}\right) = \underbrace{\ell\left(x - \frac{\ell(x)}{\ell(\hat{x})}\hat{x}\right)}_{\ell(x) - \frac{\ell(x)}{\cancel{\ell(\hat{x})}} \cdot \cancel{\ell(\hat{x})} = 0} + \ell\left(\underbrace{\frac{\ell(x)}{\ell(\hat{x})}}_{\lambda}\hat{x}\right)$$

$$= \lambda \cdot \ell(\hat{x}) \cdot \langle \hat{x}, \hat{x} \rangle = \lambda \cdot \langle \overline{\ell(\hat{x})}\hat{x}, \hat{x} \rangle = \langle x_\ell, \lambda \hat{x} \rangle$$

$$= \langle x_\ell, \underbrace{\lambda \hat{x} - x}_{\in \ker(\ell)} + x \rangle = \langle x_\ell, x \rangle$$

(2) Uniqueness: Assume $x_\ell, \tilde{x}_\ell \in X$ fulfil $\ell(x) = \langle x_\ell, x \rangle = \langle \tilde{x}_\ell, x \rangle$

$$\Rightarrow \langle x_\ell - \tilde{x}_\ell, x \rangle = 0 \quad \text{for all } x \in X.$$

$$\Rightarrow \langle x_\ell - \tilde{x}_\ell, x_\ell - \tilde{x}_\ell \rangle = 0 \Rightarrow x_\ell = \tilde{x}_\ell$$

(3) Operator norm:

$$\|\ell\| = \sup\{ |\ell(x)| \mid \|x\|_X = 1\} = \sup\{ \underbrace{|\langle x_\ell, x \rangle|}_{\leq \|x_\ell\|_X \cdot \|x\|_X} \mid \|x\|_X = 1\}$$

$$\leq \|x_\ell\|$$

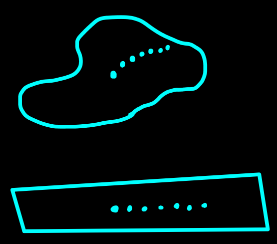
$$\|\ell\| \geq \left|\ell\left(\frac{x_\ell}{\|x_\ell\|}\right)\right| = \left|\left\langle x_\ell, \frac{x_\ell}{\|x_\ell\|}\right\rangle\right| = \|x_\ell\| \qquad \square$$

Functional analysis – part 16

Compactness

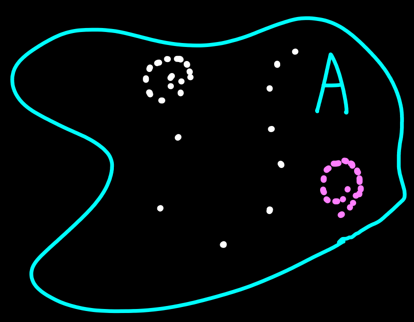
$\mathbb{R}^n \supseteq A$

$$A \text{ is compact} \overset{\uparrow \text{ only in } \mathbb{R}^n \text{ or } \mathbb{C}^n}{=} \begin{cases} \bullet\ A \text{ is closed} \\ \bullet\ A \text{ is bounded} \end{cases}$$



Definition: Let (X,d) be a metric space. $A \subseteq X$ is called (sequentially) compact if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ one finds a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with

$$\tilde{x} := \lim_{k\to\infty} x_{n_k} \in A$$



Examples: (a) $(\mathbb{R}, d_{eucl.})$, $A = [0,1]$ compact by Bolzano-Weierstrass theorem.

(b) $(\mathbb{R}, d_{discr.})$, $A = [0,1]$ not compact because:

The sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ with $x_n = \frac{1}{n}$ satisfies

$$d_{discr.}(x_n, x_m) = 1 \quad \text{for all } n,m \in \mathbb{N} \text{ with } n \neq m.$$

$\Longrightarrow$ no convergent subsequence

Proposition: Let (X,d) be a metric space and $A \subseteq X$ compact. Then A is closed and bounded. There is an $x \in X$ and an $\varepsilon > 0$ such that $B_\varepsilon(x) \supseteq A$

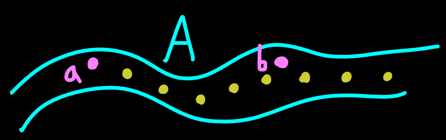
<u>Proof:</u> Let $A \subseteq X$ be compact.

(1) Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ be convergent with limit $\tilde{x} \in X$.

$\overset{\text{compact}}{\Longrightarrow}$ There is a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit $\tilde{\tilde{x}} \in A$

$\overset{\text{limit unique}}{\Longrightarrow}$ $\tilde{x} = \tilde{\tilde{x}} \in A \quad \Rightarrow \quad A$ is closed

(2) Contraposition: A is not bounded



$\Rightarrow$ For given $a \in A$, there are $x_n \in A$ with $d(a, x_n) > n$.

$\Rightarrow$ For any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and any point $b \in A$:

$$n_k < d(a, x_{n_k}) \leq d(a, b) + d(b, x_{n_k})$$

$$\Rightarrow \quad n_k - d(a, b) \leq d(b, x_{n_k})$$

$\Rightarrow \quad d(b, x_{n_k}) \;\cancel{\xrightarrow{k\to\infty}}\; 0$ for all $b \in A \quad \Rightarrow \quad A$ not compact

Functional analysis – part 17

Arzelà-Ascoli theorem

Example: (a) $(X, \|\cdot\|)$ normed space with $\dim(X) < \infty$ (always Banach space)

$A \subseteq X$: A compact $\iff$ A closed + bounded

(b) $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ for $p \in [1,\infty)$ (Banach space)

$A := \{ x \in \ell^p(\mathbb{N}) \mid \|x\|_p \leq 1 \}$ closed + bounded

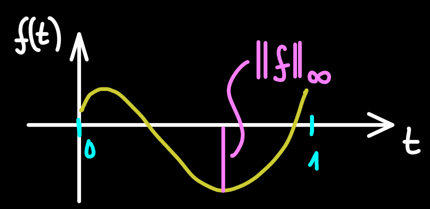
$$\left.\begin{array}{l} e_1 := (1,0,0,0,\dots) \in A \\ e_2 := (0,1,0,0,\dots) \in A \\ e_3 := (0,0,1,0,\dots) \in A \\ \vdots \qquad\qquad \vdots \end{array}\right\} \quad (e_n)_{n\in\mathbb{N}} \subseteq A$$

$$\|e_n - e_m\|_p \overset{n \neq m}{=} \sqrt[p]{|1|^p + |-1|^p} = \sqrt[p]{2}$$

$\Rightarrow$ no convergent subsequence

Continuous functions: $(C([0,1]), \|\cdot\|_\infty)$, $\|f\|_\infty := \sup\{ |f(t)| \mid t \in [0,1] \}$

↳ Banach space



f is called uniformly continuous: (Using ε-δ-characterisation)

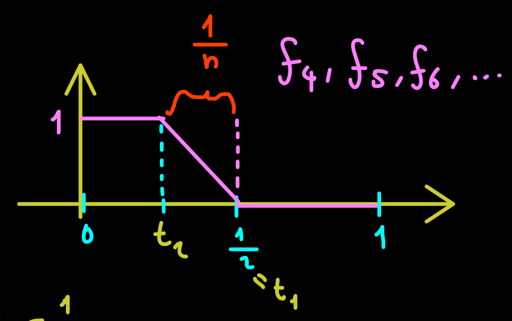
$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{t_1,t_2\in[0,1]}{\forall} : \quad |t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon$$

$A \subseteq C([0,1])$ is called uniformly equicontinuous:

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{t_1,t_2\in[0,1]}{\forall} \quad \underset{f\in A}{\forall} : \quad |t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon$$

or equivalently $\quad \sup_{f\in A} |f(t_1) - f(t_2)| \xrightarrow{|t_1 - t_2| \to 0} 0$

Examples: (a) $A := \{ f \in C([0,1]) \mid \|f\|_\infty \leq 1 \}$



$$\sup_{f \in A} |f(t_1) - f(t_2)| \geq |f_n(t_1) - f_n(t_2)| = 1 \quad \text{for } t_1 = \tfrac{1}{2},\ t_2 = \tfrac{1}{2} - \tfrac{1}{n} \quad (\text{for } n \geq 4)$$

$\Rightarrow$ A is <u>not</u> equicontinuous!

(b) $A := \{ f \in C([0,1]) \mid f \text{ continuously differentiable},\ \|f'\|_\infty \leq 2 \}$

$$|f(t_1) - f(t_2)| \overset{\text{mean value theorem}}{\leq} |f'(\xi)| \cdot |t_1 - t_2| \leq 2 \cdot |t_1 - t_2|$$

$$\sup_{f \in A} |f(t_1) - f(t_2)| \leq 2 \cdot |t_1 - t_2| \xrightarrow{|t_1 - t_2| \to 0} 0 \quad \Rightarrow \quad A \text{ is uniformly equicontinuous}$$

Arzelà-Ascoli theorem: For $(C([0,1]), \|\cdot\|_\infty)$ holds: Could be any <u>compact</u> metric space

$$A \subseteq C([0,1]) \text{ compact} \iff A \text{ is } \begin{cases} \text{closed } + \\ \text{bounded } + \\ \text{uniformly equicontinuous} \end{cases}$$

Functional analysis – part 18

Compact operators:

$T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ linear (standard norm)

$\Rightarrow$ T is continuous / bounded

$\Rightarrow$ $T[\mathcal{B}_1(0)] \subseteq \mathbb{F}^m$ bounded

$\Rightarrow$ $\overline{T[\mathcal{B}_1(0)]} \subseteq \mathbb{F}^m$ compact

However: $I: \ell^p(\mathbb{N}) \longrightarrow \ell^p(\mathbb{N})$, $p \in [1,\infty)$,

$x \longmapsto x$ $\Rightarrow$ $\overline{I[\mathcal{B}_1(0)]} = \overline{\mathcal{B}_1(0)}$ not compact

closed unit ball in $\ell^p(\mathbb{N})$

Definition: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces. A bounded linear operator $T: X \to Y$ is called compact if

$\overline{T[\mathcal{B}_1(0)]} \subseteq Y$ is a compact set.

Example: Integral operator $T_k: C([0,1]) \longrightarrow C([0,1])$ for $k \in C([0,1] \times [0,1])$

with supremum norm $\|\cdot\|_\infty$

$$(T_k f)(s) := \int_0^1 k(s,t)\, f(t)\, dt$$

Fact: k is uniformely continuous:

$$\underbrace{\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{(s_1,t_1),(s_2,t_2)} \ \|(s_1,t_1)-(s_2,t_2)\| < \delta \Rightarrow |k(s_1,t_1)-k(s_2,t_2)| < \varepsilon}$$

For $\varepsilon > 0$, choose $\delta > 0$ such that . Therefore for $s_1, s_2 \in [0,1]$ with $|s_1 - s_2| < \delta$:

$$\left|(T_k f)(s_1) - (T_k f)(s_2)\right| = \left|\int_0^1 \left(k(s_1,t)\, f(t) - k(s_2,t)\, f(t)\right) dt\right|$$

$$\leq \int_0^1 \underbrace{\left|k(s_1,t) - k(s_2,t)\right|}_{<\varepsilon} \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty} \, dt < \varepsilon \cdot \|f\|_\infty$$

$A := T_k[\mathcal{B}_1(0)]$. We have:

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{s_1, s_2 \in [0,1]}{\forall} \quad \underset{g \in A}{\forall} : |s_1 - s_2| < \delta \Rightarrow |g(s_1) - g(s_2)| < \varepsilon$$

$\Rightarrow T_k[\mathcal{B}_1(0)]$ is uniformely equicontinuous

Boundedness: $\|T_k\| = \sup\left\{ \|T_k f\|_\infty \;\middle|\; \|f\|_\infty = 1 \right\}$

$$= \sup\left\{ \sup_{s\in[0,1]} \left|\int_0^1 k(s,t) f(t)\, dt\right| \;\middle|\; \|f\|_\infty = 1 \right\}$$

$$\leq \sup\left\{ \sup_{s\in[0,1]} \int_0^1 |k(s,t)| \underbrace{|f(t)|}_{\leq \|f\|_\infty} dt \;\middle|\; \|f\|_\infty = 1 \right\}$$

$$\leq \sup_{s\in[0,1]} \int_0^1 |k(s,t)|\, dt \leq \|k\|_\infty$$

$\Rightarrow$ By Arzelà-Ascoli: $\overline{T_k[\mathcal{B}_1(0)]}$ is compact $\Rightarrow T_k$ compact operator

Functional analysis – part 19

Hölder's inequality (for $\mathbb{F}^n$ and $p \in (1,\infty)$)

↳ $p' \in (1, \infty)$ Hölder conjugate

For $x \in \mathbb{F}^n$:

$$\|x\|_q := \left(\sum_{j=1}^{n} |x_j|^q \right)^{\frac{1}{q}} , \quad q \in [1,\infty) \qquad \boxed{\frac{1}{p} + \frac{1}{p'} = 1}$$

For $x, y \in \mathbb{F}^n$ write:

$$xy := \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{pmatrix}$$

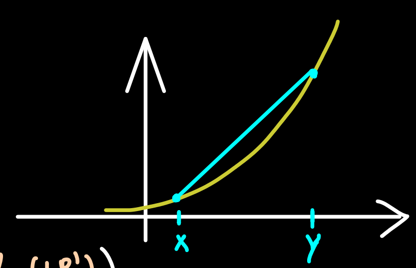
Then: $\|xy\|_1 \leq \|x\|_p \cdot \|y\|_{p'}$ for all $x, y \in \mathbb{F}^n$

Young's inequality:

$a, b > 0 \Rightarrow a \cdot b \leq \dfrac{a^p}{p} + \dfrac{b^{p'}}{p'}$

Proof: $f: x \mapsto e^x$ is convex: $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$



$$a \cdot b = f(\log(a) + \log(b)) = f(\underbrace{\lambda}_{\frac{1}{p}} \underbrace{x}_{\log(a^p)} + \underbrace{(1-\lambda)}_{\frac{1}{p'}} \underbrace{y}_{\log(b^{p'})}) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$= \frac{1}{p} f(\log(a^p)) + \frac{1}{p'} f(\log(b^{p'})) = \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$$

Proof of Hölder's inequality:

1st case: $x = 0$ or $y = 0$

2nd case:

$$\frac{1}{\|x\|_p \cdot \|y\|_{p'}} \|xy\|_1 = \frac{1}{\|x\|_p \cdot \|y\|_{p'}} \sum_{j=1}^{n} |x_j y_j| = \sum_{j=1}^{n} \overbrace{\frac{|x_j|}{\|x\|_p}}^{a} \cdot \overbrace{\frac{|y_j|}{\|y\|_{p'}}}^{b}$$

$$\leq \sum_{j=1}^{n} \frac{1}{p} \cdot \frac{|x_j|^p}{\|x\|_p^p} + \sum_{j=1}^{n} \frac{1}{p'} \frac{|y_j|^{p'}}{\|y\|_{p'}^{p'}} = \frac{1}{p} + \frac{1}{p'} = 1$$

Functional analysis – part 20

Minkowski's inequality: Δ-inequality for $\|\cdot\|_p$ in $\ell^p(\mathbb{N})$:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \text{for all } x,y \in \ell^p(\mathbb{N}), \quad p \in [1,\infty)$$

Proof: For $p=1$: $\|x+y\|_1 = \sum_{j=1}^{\infty} \underbrace{|x_j+y_j|}_{\leq |x_j|+|y_j|} \leq \|x\|_1 + \|y\|_1$

For $p \in (1,\infty)$: Hölder conjugate $p' \in (1,\infty)$ $\quad \boxed{\frac{1}{p} + \frac{1}{p'} = 1}$

$$\frac{p}{p-1} = p'$$

$$\|x+y\|_p^p = \sum_{j=1}^{\infty} |x_j+y_j|^p = \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{|x_j+y_j|}_{\leq |x_j|+|y_j|}{}^p = (*)$$

$$(**) \quad \big(|x_j|+|y_j|\big)^p = \big(|x_j|+|y_j|\big)\big(|x_j|+|y_j|\big)^{p-1} = \underbrace{|x_j|}_{a_j}\underbrace{\big(|x_j|+|y_j|\big)^{p-1}}_{b_j} + \underbrace{|y_j|}_{c_j}\big(|x_j|+|y_j|\big)^{p-1} \rightsquigarrow a,b,c \in \mathbb{F}^n$$

Hölder: $\|ab\|_1 \leq \|a\|_p \cdot \underbrace{\|b\|_{p'}}_{= \left(\sum_{j=1}^{n} \left|(|x_j|+|y_j|)^{p-1}\right|^{p'}\right)^{\frac{1}{p'}} = \left(\sum_{j=1}^{n} (|x_j|+|y_j|)^p\right)^{\frac{1}{p'}}}$

$$(***) \quad \sum_{j=1}^{n}\big(|x_j|+|y_j|\big)^p \leq \|a\|_p \cdot \|b\|_{p'} + \|c\|_p \cdot \|b\|_{p'} = \big(\|a\|_p + \|c\|_p\big) \cdot \left(\sum_{j=1}^{n}\big(|x_j|+|y_j|\big)^p\right)^{\frac{1}{p'}}$$

$$\Rightarrow \left(\sum_{j=1}^{n}\big(|x_j|+|y_j|\big)^p\right)^{\overbrace{1-\frac{1}{p'}}^{\frac{1}{p}}} \leq \left(\sum_{j=1}^{n}|x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n}|y_j|^p\right)^{\frac{1}{p}}$$

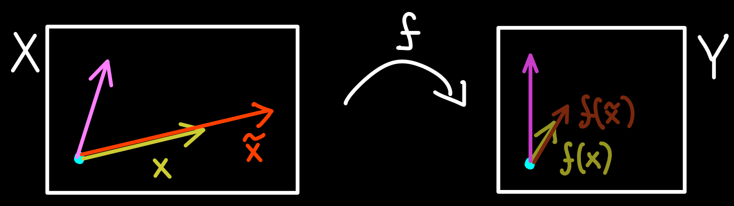
$$\xrightarrow[+\,(*)]{n\to\infty} \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Functional analysis – part 21

Isomorphisms?

Homomorphism: map that preserves structures

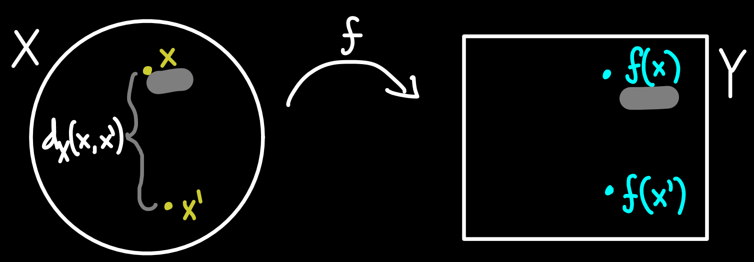
Example: (a) Let X, Y be vector spaces and $f: X \to Y$ be a map.



$$\left.\begin{array}{r} f(\lambda x) = \lambda \cdot f(x) \\ f(x + x') = f(x) + f(x') \end{array}\right\} \text{linear}$$

homomorphism = linear map

(b) Let (X, d_X), (Y, d_Y) be two metric spaces and $f: X \to Y$ be a map.



$$d_Y(f(x), f(x')) \leq d_X(x, x') \quad (*)$$

homomorphism = map that satisfies (*)

isomorphism = homomorphism + bijective + inverse map is also homomorphism

Isomorphism for Banach spaces X, Y:

$f: X \to Y$ with: linear + bijective + $\|f(x)\|_Y = \|x\|_X$

(often called isometric isomorphism)

Example: (a) $S_R: \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$

$\Rightarrow$ linear, $\|S_R x\|_p = \|x\|_p$ not surjective $\Rightarrow$ not an isomorphism

(b) $S: \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$, $(\dots, x_{-1}, x_0, x_1, x_2, \dots) \mapsto (\dots, x_{-2}, x_{-1}, x_0, x_1, \dots)$

index $\dots$ -1 0 1 2 $\dots$ index $\dots$ -1 0 1 2 $\dots$

$\Rightarrow$ linear, $\|Sx\|_p = \|x\|_p$ and bijective $\Rightarrow$ isomorphism

Functional analysis – part 22

Dual spaces: X normed space

X' normed space

$$X' := \{ \ell : X \to \mathbb{F} \mid \ell \text{ linear + bounded} \}$$

Recall the Riesz representation theorem: X Hilbert space. Then: $X' \overset{\text{isometric isomorphism}}{\longleftrightarrow} X$

Proposition: Let X be a normed space. Then $(X', \|\cdot\|_{X\to\mathbb{F}})$ is a Banach space.

Proof: Let $(\ell_k)_{k\in\mathbb{N}} \subseteq X'$ be a Cauchy sequence:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n,m \geq N : \quad \|\ell_n - \ell_m\|_{X\to\mathbb{F}} < \varepsilon$$

$$\geq \frac{1}{\|x\|_X} |\ell_n(x) - \ell_m(x)| \quad \text{for } x \in X,\ x \neq 0.$$

$\Rightarrow (\ell_k(x))_{k\in\mathbb{N}} \subseteq \mathbb{F}$ is Cauchy sequence for all $x \in X$.

$$\Rightarrow \quad \ell(x) := \lim_{k\to\infty} \ell_k(x), \quad \ell : X \to \mathbb{F}$$

Show:
(1) ℓ is linear ✓
(2) ℓ is bounded ✓
(3) $\|\ell_k - \ell\|_{X\to\mathbb{F}} \xrightarrow{k\to\infty} 0$ ✓

For (2): $\|\ell_n\|_{X\to\mathbb{F}} \leq \underbrace{\|\ell_n - \ell_N\|_{X\to\mathbb{F}}}_{<\varepsilon} + \underbrace{\|\ell_N\|_{X\to\mathbb{F}}}_{=:C} \leq C + \varepsilon$ for all $n \geq N$

$$\Rightarrow \quad |\ell(x)| = \left|\lim_{k\to\infty} \ell_k(x)\right| = \lim_{k\to\infty} |\ell_k(x)| \leq \lim_{k\to\infty} \underbrace{\|\ell_k\|_{X\to\mathbb{F}}}_{\leq \tilde{C}} \|x\|_X$$

$$\Rightarrow \quad \|\ell\|_{X\to\mathbb{F}} \leq \tilde{C} < \infty$$

For (3): For $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n,m \geq N$:

$$\frac{1}{\|x\|_X} |\ell_n(x) - \ell_m(x)| < \varepsilon$$

$$\Rightarrow \quad \sup_{\substack{x\in X\\ x\neq 0}} \frac{1}{\|x\|_X} \Big|\ell_n(x) - \overbrace{\lim_{m\to\infty} \ell_m(x)}^{\ell(x)}\Big| \leq \varepsilon \quad \Rightarrow \quad \|\ell_n - \ell\|_{X\to\mathbb{F}} \leq \varepsilon$$

Functional analysis – part 23

Dual space: X normed space

$$X' := \{ \ell : X \to \mathbb{F} \mid \ell \text{ linear + bounded} \}$$

Example: $X = \ell^p(\mathbb{N})$ for $p \in (1, \infty)$

$$X' \cong \ell^{p'}(\mathbb{N}) \quad \text{where} \quad p' \in (1,\infty) \text{ Hölder conjugate} \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$$

there is an isometric isomorphism

$$T : \ell^{p'}(\mathbb{N}) \longrightarrow \left(\ell^p(\mathbb{N})\right)'$$

$$(Tx)(y) := \sum_{k=1}^{\infty} x_k \cdot y_k \quad \text{or} \quad x \mapsto \langle \overline{x}, \cdot \rangle_{\ell^2(\mathbb{N})}$$

To show:
(1) T is well-defined ✓
(2) T is linear ✓
(3) T is bounded ✓
(4) T surjective
(5) $\|Tx\| = \|x\|$ for all $x \in \ell^{p'}(\mathbb{N})$ (isometric)

Proof: (1) $|(Tx)(y)| \leq \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} |y_k \cdot x_k| \overset{\text{Hölder}}{\leq} \|y\|_p \cdot \|x\|_{p'} < \infty$

$\Rightarrow$ Tx is linear and bounded for all $x \in \ell^{p'}(\mathbb{N})$

(2) T is linear.

(3) $$\|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} = \sup\{ \underbrace{|(Tx)(y)|}_{\leq \|y\|_p \cdot \|x\|_{p'}} \mid \|y\|_p = 1 \} \leq \|x\|_{p'}$$

$$\boxed{T: \ell^{p'}(\mathbb{N}) \longrightarrow \left(\ell^{p}(\mathbb{N})\right)'} \qquad \Rightarrow \quad \|T\| \leq 1$$

(4) Let $y' \in \left(\ell^{p}(\mathbb{N})\right)'$ and $e_k = (0, 0, \dots, 0, \overset{k^{th}\text{ position}}{1}, 0, \dots)$.

Define: $x_k := y'(e_k)$ and $x := (x_k)_{k \in \mathbb{N}}$

Question: $x \in \ell^{p'}(\mathbb{N})$ and $Tx = y'$?

$$\sum_{k=1}^{n} |x_k|^{p'} = \sum_{k=1}^{n} x_k \cdot t_k \qquad t_k := \begin{cases} \dfrac{|x_k|^{p'}}{x_k}, & x_k \neq 0 \\ 0, & x_k = 0 \end{cases}$$

$$= \sum_{k=1}^{n} t_k \cdot y'(e_k) = y'\left(\sum_{k=1}^{n} t_k e_k\right)$$

$$\leq \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \cdot \underbrace{\Big\| \overbrace{\sum_{k=1}^{n} t_k e_k}^{=(t_1, t_2, \dots, t_n, 0, 0, \dots)} \Big\|_p}_{\left(\sum_{k=1}^{n} |t_k|^p\right)^{\frac{1}{p}}} \qquad \boxed{\left(\frac{1}{p} + \frac{1}{p'} = 1\right)}$$

$$|t_k|^p = \left(\frac{|x_k|^{p'}}{|x_k|}\right)^p = |x_k|^{\overbrace{(p'-1)\cdot p}^{p'}}$$

$$= \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \left(\sum_{k=1}^{n} |x_k|^{p'}\right)^{\frac{1}{p}}$$

$$\overset{n \to \infty}{\Longrightarrow} \|x\|_{p'} \leq \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \quad \Rightarrow \quad x \in \ell^{p'}(\mathbb{N}) \checkmark$$

For $y \in \ell^p(\mathbb{N})$: $(Tx - y')(y) = (Tx - y')\left(\lim_{n \to \infty} \sum_{k=1}^{n} y_k e_k\right)$

$$\boxed{(Tx)(y) := \sum_{j=1}^{\infty} x_j \cdot y_j}$$

$$\overset{\text{continuity}}{=} \lim_{n \to \infty} (Tx - y')\left(\sum_{k=1}^{n} y_k e_k\right)$$

$$\overset{\text{linearity}}{=} \lim_{n \to \infty} \sum_{k=1}^{n} y_k \cdot (Tx - y')(e_k) = 0 \qquad \text{surjective} \checkmark$$

(5) $\|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} \underset{=}{\leq} \|x\|_{p'} \underset{=}{\leq} \|y'\|_{\ell^p(\mathbb{N}) \to \mathbb{F}} = \|Tx\|_{\ell^p(\mathbb{N}) \to \mathbb{F}}$ isometry $\checkmark$

Functional analysis – part 24

Uniform boundedness principle (Banach-Steinhaus theorem)

X, Y normed spaces, X Banach space.

$$\mathcal{B}(X,Y) := \{ T : X \to Y \mid T \text{ linear + bounded} \}$$

Theorem: For every subset $\mathcal{M} \subseteq \mathcal{B}(X,Y)$ holds:

$\mathcal{M}$ is bounded pointwise on X $\Longleftrightarrow$ $\mathcal{M}$ is uniformely bounded

More concretely: $\forall_{x \in X} \ \exists_{C_x \geq 0} \ \forall_{T \in \mathcal{M}} \ \|Tx\|_Y \leq C_x \Longleftrightarrow \exists_{C \geq 0} \ \forall_{T \in \mathcal{M}} \ \|T\|_{X \to Y} \leq C$

Proposition: X, Y normed spaces, X Banach space.

Let $T_n \in \mathcal{B}(X,Y)$ for all $n \in \mathbb{N}$ with $\lim\limits_{n \to \infty} T_n x$ exists for all $x \in X$.

Then: $T : X \to Y$ defined by $Tx := \lim\limits_{n \to \infty} T_n x$ is linear and bounded.

Proof: $\mathcal{M} := \{ T_n \mid n \in \mathbb{N} \}$ is bounded pointwise on X $\overset{\text{Banach–Steinhaus}}{\Longrightarrow}$ There is a $C \geq 0$ with $\|T_n\| \leq C$ for all n

$$\Rightarrow \|T\|_{X \to Y} = \sup \{ \underbrace{\|Tx\|_Y} \mid \|x\|_X = 1 \} \leq C$$

$$\|\lim_{n \to \infty} T_n x\|_Y = \lim_{n \to \infty} \|T_n x\|_Y \leq C$$

Functional analysis – part 25

Hahn-Banach theorem $(X, \|\cdot\|_X)$ normed space $\rightsquigarrow$ $(X', \|\cdot\|_{X'})$

$U \subseteq X$ subspace, $u' : U \longrightarrow \mathbb{F}$ continuous linear functional

Then: There exists $x' : X \longrightarrow \mathbb{F}$ continuous linear functional

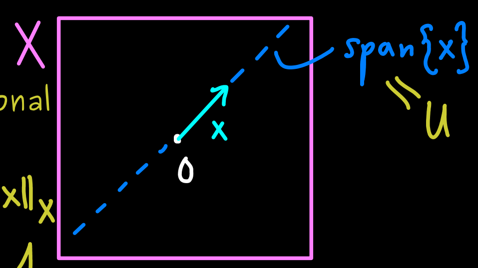
with $x'(u) = u'(u)$ for all $u \in U$,

$\|x'\|_{X'} = \|u'\|_{U'}$.

Applications: $(X, \|\cdot\|_X)$ normed space

(a) For all $x \in X$, $x \neq 0$, there is an $x' \in X'$ with $\|x'\|_{X'} = 1$ and $x'(x) = \|x\|_X$.

Proof: Define $u' : U \longrightarrow \mathbb{F}$, $\lambda \cdot x \longmapsto \lambda \cdot \|x\|_X$ continuous linear functional



$\overset{\text{Hahn-Banach}}{\Longrightarrow}$ $x' : X \longrightarrow \mathbb{F}$ with $x'(x) = u'(x) = \|x\|_X$

$\|x'\|_{X'} = \|u'\|_{U'} = 1$

(b) X' separates the points of X: For $x_1, x_2 \in X$, $x_1 \neq x_2$,

there is an $x' \in X'$ with $x'(x_1) \neq x'(x_2)$

Proof: $x := x_2 - x_1$ $\overset{(a)}{\Longrightarrow}$ $\underbrace{x'(x)}_{= x'(x_2) - x'(x_1)} = \|x\|_X \neq 0$ $\Longrightarrow$ $x'(x_1) \neq x'(x_2)$

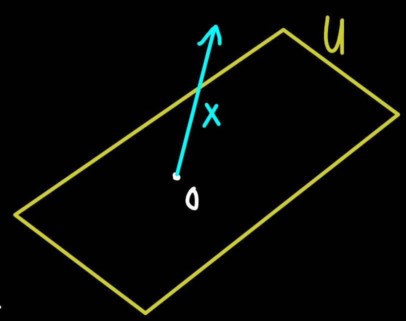
(c) For all $x \in X$: $\|x\|_X = \sup\{ |x'(x)| \mid x' \in X', \|x'\| = 1 \}$

Proof:
$$\|x'\|_{X'} \geq \frac{|x'(x)|}{\|x\|_X} \Rightarrow 1 = \sup_{\|x'\|=1} \|x'\|_{X'} \geq \sup_{\|x'\|=1} \frac{|x'(x)|}{\|x\|_X}$$
$$\Rightarrow \|x\|_X \geq \sup_{\|x'\|=1} |x'(x)|$$

Use (a): $\|x\|_X \leq \sup_{\|x'\|=1} |x'(x)|$

(d) Let $U \subseteq X$ be a closed subspace, $x \in X$ with $x \notin U$.

Then there exists $x' \in X'$ with $x'|_U = 0$ and $x'(x) \neq 0$.



Proof: $X/U := \{ [z] \mid z \in X \}$, $[z] := \{ z + u \mid u \in U \}$

$\|[z]\|_{X/U} := \inf_{u \in U} \|z + u\|_X$ $\rightsquigarrow$ $(X/U, \|\cdot\|_{X/U})$ normed space

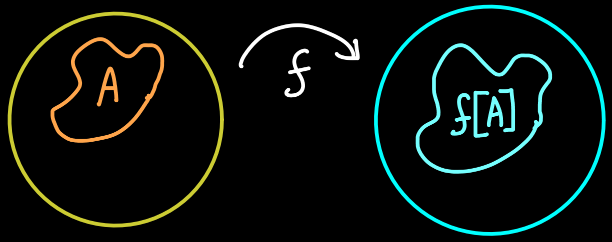
$\overset{(a)}{\Longrightarrow}$ There is a $y' \in (X/U)'$ with $y'([x]) \neq 0$.

Define $x' \in X'$ by $x'(z) := y'([z])$ for $z \in X$.

Functional analysis – part 26

Open mapping theorem (Banach–Schauder theorem)

What is an open map?



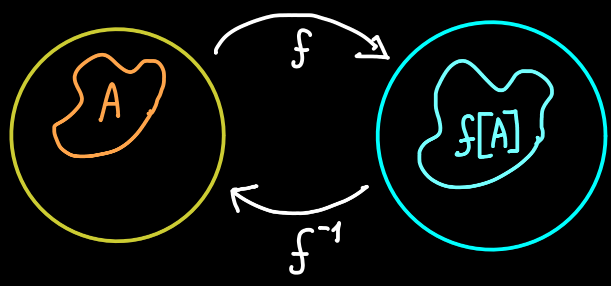
Let (X, d_X), (Y, d_Y) be two metric spaces.

$f: X \to Y$ is called <u>open</u> if

$$A \subseteq X \text{ open in } X \Rightarrow f[A] \subseteq Y \text{ open in } Y$$

<u>General example:</u> If $f: X \to Y$ is bijective and $f^{-1}: Y \to X$ is continuous, then:

$f: X \to Y$ is an open map



Continuity of f^{-1}: $A \subseteq X$ open in $X \Rightarrow \underbrace{(f^{-1})^{-1}[A]}_{f[A]} \subseteq Y$ open in Y

<u>Examples:</u> (a) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ open

(b) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ not open $\quad A = (-2, 2) \rightsquigarrow f[A] = [0, 4)$

<u>Open Mapping Theorem:</u> Let X, Y be Banach spaces. For $T \in \mathcal{B}(X, Y)$ holds:

$$T \text{ surjective} \iff T \text{ open map}$$

Functional analysis – part 27

Bounded inverse theorem: X, Y Banach spaces, $T \in \mathcal{B}(X,Y)$.

Then: T bijective $\Rightarrow$ $T^{-1} \in \mathcal{B}(Y,X)$ (It's continuous)

Proof: T bijective $\underset{\text{open mapping theorem}}{\Rightarrow}$ T open map $\Rightarrow$ T^{-1} continuous $\square$

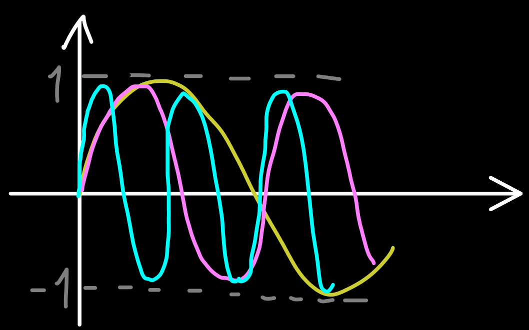
Counterexample: $X = \left(C([0,1]), \|\cdot\|_\infty\right)$, $Y = \left(\{ f \in C^1([0,1]) \mid f(0) = 0 \}, \|\cdot\|_\infty\right)$ $\longrightarrow$ not complete

$$(Tf)(t) = \int_0^t f(s)\, ds$$ linear and bounded and bijective

$$\|Tf\|_\infty = \sup_{t \in [0,1]} \left| \int_0^t f(s)\, ds \right| \leq \|f\|_\infty \Rightarrow \|T\|_{X \to Y} \leq 1$$

Take $f_k(t) = \sin(kt)$

$$(Tf_k)(t) = \underbrace{\frac{1}{k}\left(1 - \cos(kt)\right)}_{g_k(t)}$$



$$T^{-1} g_k = f_k \Rightarrow \|T^{-1}\|_{Y \to X} \geq \frac{\|T^{-1} g_k\|_\infty}{\|g_k\|_\infty} = \frac{\overbrace{\|f_k\|_\infty}^{\leq 1}}{\underbrace{\|g_k\|_\infty}_{\leq \frac{2}{k}}} \geq \frac{k}{2} \xrightarrow{k \to \infty} \infty$$

$\Rightarrow$ T^{-1} not continuous

Functional analysis – part 28

Spectrum for bounded linear operators

Recall: $A \in \mathbb{C}^{n \times n}$ matrix with n rows and n columns.

$\lambda \in \mathbb{C}$ is called an eigenvalue of A if:

$$\exists x \in \mathbb{C}^n \setminus \{0\} : \quad Ax = \lambda x$$

$$\Leftrightarrow \exists x \in \mathbb{C}^n \setminus \{0\} : \quad (A - \lambda I)x = 0$$

$$\Leftrightarrow \operatorname{Ker}(A - \lambda I) \neq \{0\} \qquad \Leftrightarrow \quad \text{map } x \mapsto (A - \lambda I)x \text{ not injective}$$

Rank-nullity theorem: For any matrix $M \in \mathbb{C}^{m \times n}$:

$$\dim(\operatorname{Ran}(M)) + \dim(\operatorname{Ker}(M)) = n$$

Now: Let X be a complex Banach space and $T: X \to X$ be a bounded linear operator.

Definition: The spectrum of T is defined by: $\sigma(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not bijective} \}$

The resolvent set of T is defined by: $\rho(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ bijective and } (T - \lambda I)^{-1} \text{ bounded} \}$

bounded inverse theorem

$$\Longrightarrow \qquad \sigma(T) = \mathbb{C} \setminus \rho(T)$$

We have the disjoint union: $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$

point spectrum $\quad \sigma_p(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ not injective} \}$

continuous spectrum $\quad \sigma_c(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective with } \overline{\operatorname{Ran}(T - \lambda I)} = X \}$

residual spectrum $\quad \sigma_r(T) := \{ \lambda \in \mathbb{C} \mid (T - \lambda I) \text{ injective but not surjective with } \overline{\operatorname{Ran}(T - \lambda I)} \neq X \}$

Functional analysis – part 29

Let X be a complex Banach space and
$T: X \longrightarrow X$ be a bounded linear operator.

$$\lambda \in \sigma(T) \iff (T - \lambda) \quad \text{not invertible}$$

Finite-dimensional example: $X = \mathbb{C}^n$, $\quad Tx = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$

$$\Longrightarrow \sigma(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sigma_p(T)$$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors

Infinite-dimensional example: $X = \ell^p(\mathbb{N})$, $\quad p \in [1, \infty)$

$$Tx = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \end{pmatrix}$$

Formally: For $\lambda_1, \lambda_2, \dots \in \mathbb{C}$ with $\sup_{j \in \mathbb{N}} |\lambda_j| < \infty$, define: $T: \ell^p(\mathbb{N}) \longrightarrow \ell^p(\mathbb{N})$

$$(Tx)_j := \lambda_j x_j$$

- $e_1 = (1, 0, 0, \dots)$ is an eigenvector with eigenvalue λ_1
- $e_2 = (0, 1, 0, \dots)$ is an eigenvector with eigenvalue λ_2
- $\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\Longrightarrow \sigma(T) \supseteq \{\lambda_1, \lambda_2, \dots\} = \sigma_p(T)$$

Let $\mu \in \mathbb{C}$ be a number with $\mu \notin \{\lambda_1, \lambda_2, \dots\}$ but $\mu \in \overline{\{\lambda_1, \lambda_2, \dots\}}$.

e.g. $\lambda_j = \frac{1}{j}$ then $\mu = 0$

$\Longrightarrow$ $T-\mu$ is injective

<u>Show:</u> $T-\mu$ is not surjective

<u>Proof:</u> Assume $T-\mu$ is surjective $\Longrightarrow$ $T-\mu$ is bijective $\overset{\text{bounded inverse theorem}}{\Longrightarrow}$ $(T-\mu)^{-1}$ bounded

$$\Longrightarrow \|(T-\mu)^{-1}\| \geq \|(T-\mu)^{-1} e_j\|_{\ell^p(\mathbb{N})} = \|(\lambda_j-\mu)^{-1} e_j\|_{\ell^p(\mathbb{N})} = |(\lambda_j - \mu)^{-1}|$$

$$= \frac{1}{|\lambda_j - \mu|} \xrightarrow{\text{for a subsequence}} \infty \quad ↯$$

<u>Result:</u> $$\sigma(T) = \underbrace{\{\lambda_1, \lambda_2, \dots\}}_{\sigma_p(T)} \cup \underbrace{\left\{\mu \in \mathbb{C} \;\middle|\; \mu \notin \{\lambda_1, \lambda_2, \dots\} \wedge \mu \in \overline{\{\lambda_1, \lambda_2, \dots\}}\right\}}_{\sigma_c(T)\ \cancel{\cup\ \sigma_r(T)}}$$

$p \in [1, \infty)$

Functional analysis – part 30

X complex Banach space

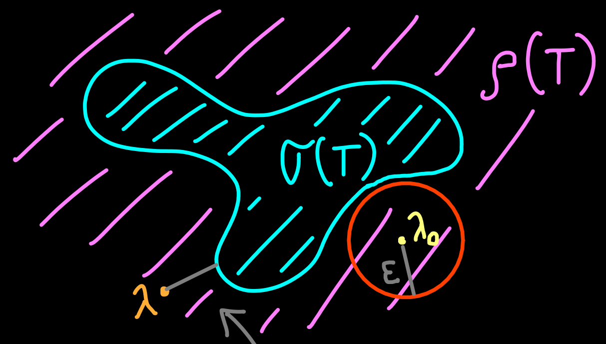
$T : X \to X$ bounded linear operator

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid (T-\lambda) \text{ not invertible}\}$$

$$\rho(T) := \{\lambda \in \mathbb{C} \mid (T-\lambda) \text{ invertible}\}$$

Proposition: (a) $\rho(T)$ is an open set

$\sigma(T)$ is a closed set



(b) For $\lambda \in \rho(T)$: $\quad \|(T-\lambda)^{-1}\| \geq \dfrac{1}{\operatorname{dist}(\lambda, \sigma(T))}$

(c) The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$\lambda \longmapsto (T-\lambda)^{-1}$ is analytical.

Locally, it can be expressed as a Taylor series.

Proof: Choose $\lambda_0 \in \rho(T)$ and set $C := \|(T-\lambda_0)^{-1}\|$, $\varepsilon := \dfrac{1}{C}$

Let's take any $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon$.

Calculate: $T - \lambda = (T-\lambda_0) - (\lambda-\lambda_0) = (T-\lambda_0)\Big(I - \underbrace{(\lambda-\lambda_0)\cdot(T-\lambda_0)^{-1}}_{S}\Big)$

$\|S\| < \varepsilon \cdot C = 1$

> Neumann series: $(I - S)$ with $\|S\| < 1$ is invertible because
>
> $$(I-S)\cdot\sum_{k=0}^{n} S^k = (I - S^{n+1}) \xrightarrow{n\to\infty} I \quad \Rightarrow (I-S)^{-1} = \sum_{k=0}^{\infty} S^k$$

$\Longrightarrow$ $T-\lambda$ is invertible $\Longrightarrow$ $\lambda \in \rho(T)$ $\Longrightarrow$ $\rho(T)$ is open (a)

Also: $(T-\lambda)^{-1} = (I-S)^{-1}(T-\lambda_0)^{-1} = \sum_{k=0}^{\infty} S^k \cdot (T-\lambda_0)^{-1}$ (c)

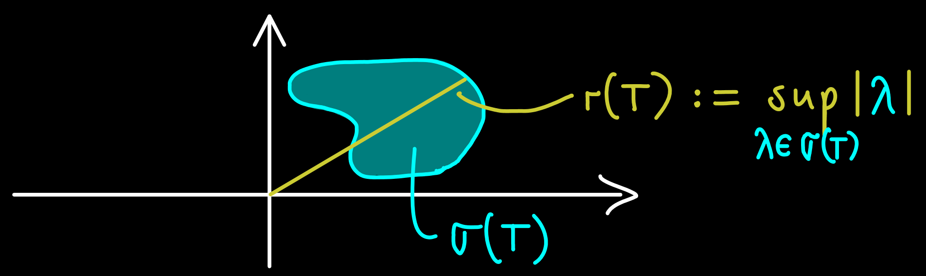
$$= \sum_{k=0}^{\infty} (\lambda-\lambda_0)^k \cdot (T-\lambda_0)^{-k} \cdot (T-\lambda_0)^{-1} = \sum_{k=0}^{\infty} (T-\lambda_0)^{-(k+1)} \cdot (\lambda-\lambda_0)^k$$

Now for $\lambda \in \sigma(T)$ $\overset{\text{above}}{\Longrightarrow}$ $|\lambda-\lambda_0| \geq \varepsilon \Longrightarrow \dfrac{1}{|\lambda-\lambda_0|} \leq C = \|(T-\lambda_0)^{-1}\|$

$$\frac{1}{\operatorname{dist}(\lambda_0, \sigma(T))} = \frac{1}{\inf\limits_{\lambda\in\sigma(T)} |\lambda-\lambda_0|} = \sup_{\lambda\in\sigma(T)} \frac{1}{|\lambda-\lambda_0|} \leq \|(T-\lambda_0)^{-1}\| \quad \text{(b)}$$

Functional analysis — part 31

Spectral radius: X complex Banach space $\quad T : X \longrightarrow X$ bounded linear operator



Theorem: X complex Banach space, $T : X \longrightarrow X$ bounded linear operator.

Then: (a) $\sigma(T) \subseteq \mathbb{C}$ is compact

(b) $X \neq \{0\} \Longrightarrow \sigma(T) \neq \emptyset$

(c) $r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{k \to \infty} \|T^k\|^{\frac{1}{k}} = \inf_{k \in \mathbb{N}} \|T^k\|^{\frac{1}{k}} \leq \|T\| < \infty$

Proof: For $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$: $\left(I - \frac{T}{\lambda} \right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k$

$$\Longrightarrow \boxed{(T - \lambda)^{-1} = -\frac{1}{\lambda} \left(I - \frac{T}{\lambda} \right)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k \qquad (*)}$$

$$\Longrightarrow \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\| \Longrightarrow \sigma(T) \text{ is bounded}$$

For (b): Assume $\sigma(T) = \emptyset \implies \rho(T) = \mathbb{C}$

Reminder: The map $\rho(T) \longrightarrow \mathcal{B}(X)$

$$\lambda \longmapsto (T-\lambda)^{-1} \quad \text{is analytic.}$$

Take any $\ell \in \mathcal{B}(X)'$: $\quad f_\ell : \mathbb{C} \longrightarrow \mathbb{C}$

$$\lambda \longmapsto \ell((T-\lambda)^{-1})$$

analytic function (holomorphic function)

We get that f_ℓ is a bounded entire function.

$$\boxed{(T-\lambda)^{-1} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{T}{\lambda}\right)^k \quad (*)}$$

For $|\lambda| \geq 2\cdot\|T\|$:

$$|f_\ell(\lambda)| \leq \|\ell\|\cdot\|(T-\lambda)^{-1}\| \leq \|\ell\| \underbrace{\frac{1}{|\lambda|}}_{\leq \frac{1}{2\|T\|}} \sum_{k=0}^{\infty} \underbrace{\left\|\frac{T}{\lambda}\right\|^k}_{\leq \frac{1}{2}}$$

$$\leq \frac{\|\ell\|}{\|T\|}$$

$$\xRightarrow{\text{Liouville's theorem}} f_\ell \text{ is constant}$$

$$f_\ell(0) = \ell(T^{-1})$$

$$\|$$

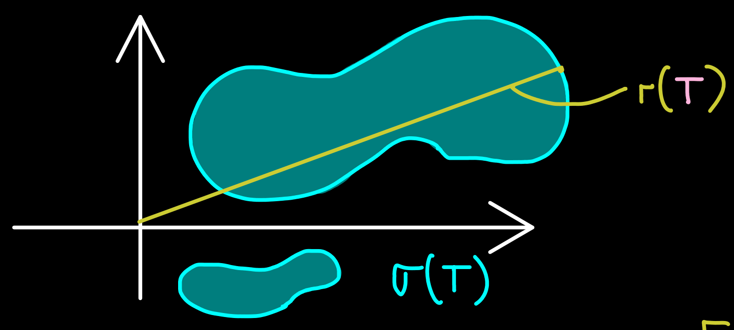
$$f_\ell(\lambda) = \ell((T-\lambda)^{-1}) = \ell\left(\sum_{k=0}^{\infty}(T)^{-(k+1)}\cdot(\lambda)^k\right)$$

$$= \sum_{k=0}^{\infty}\ell\left(T^{-(k+1)}\right)\cdot\lambda^k$$

$$\implies \ell(T^{-2}) = 0 \quad \text{for all } \ell \in \mathcal{B}(X)'$$

$$\underset{\text{(part 25)}}{\xRightarrow{\text{Hahn–Banach theorem}}} T^{-2} = 0 \implies X = \{0\}$$

Functional analysis – part 32



$T: X \to X$ bounded linear operator

$$r(T) \leq \|T\|$$

For normal operators: $r(T) = \|T\|$

X is a complex Hilbert space

Definition: Let X be a Hilbert space and $T: X \to X$ a bounded linear operator. The bounded linear operator $T^*: X \to X$ defined by

$$\langle y, Tx \rangle = \langle T^* y, x \rangle \quad \text{for all } x, y \in X$$

is called the adjoint operator of T.

Definition: Let X be a Hilbert space and $T: X \to X$ a bounded linear operator. T is called

(1) self-adjoint if $T^* = T$

(2) skew-adjoint if $T^* = -T$

(3) normal if $T^*T = TT^*$

Proposition: T normal $\Rightarrow$ $r(T) = \|T\|$

Functional analysis – part 33

Compact operator: $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces.

$T: X \to Y$ bounded linear operator is called compact if

$$\overline{T[B_1(0)]} \text{ is compact.}$$

Example: matrix $A \in \mathbb{C}^{n \times n}$ (linear operator $\mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto Ax$) $\hookrightarrow$ compact

We know: $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ finite, non-empty set

$\mathrm{Ker}(A - \lambda_j)$ eigenspaces (finite-dimensional)

Proposition: $(X, \|\cdot\|_X)$ Banach space, $T: X \to X$ compact operator. Then:

(a) $\sigma(T)$ countable set (finite is possible)

(b) $\dim(X) = \infty \Rightarrow 0 \in \sigma(T)$

(c) $\sigma(T) \setminus \{0\}$ could be empty or finite.

Otherwise: $\sigma(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$ ← no accumulation points other than 0

(d) Each $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue of T $(\lambda \in \sigma_p(T))$

with $\dim(\mathrm{Ker}(T - \lambda)) < \infty$

Example: $X = \ell^2(\mathbb{N})$, $Tx = \left(\frac{1}{j} x_j\right)_{j \in \mathbb{N}}$

$$\overline{T[B_1(0)]} \subseteq \left\{ y \in \ell^2(\mathbb{N}) \;\middle|\; |y_j| \leq \frac{1}{j} \text{ for all } j \right\}$$

Hilbert cube

$\downarrow$

compact set

$\Rightarrow$ T is a compact operator

$$T = \begin{pmatrix} \frac{1}{1} & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ & & & \frac{1}{4} & \\ & & & & \ddots \end{pmatrix}$$

$T e_k = \frac{1}{k} e_k$ (eigenvector) $\dim\left(\operatorname{Ker}\left(T - \frac{1}{k}\right)\right) = 1$

$$\sigma(T) = \left\{ \frac{1}{1}, \frac{1}{2}, \dots \right\} \cup \{0\}$$

Functional Analysis – Part 34

Spectral theorem of compact operators

Let X be a complex Hilbert space and $T: X \to X$ be a compact operator.
Assume that T is self-adjoint $(T^* = T)$ or normal $(T^*T = TT^*)$.

Then there is an orthonormal system $\{e_i \mid i \in I\}$ with $I \subseteq \mathbb{N}$
and a sequence $(\lambda_i)_{i \in I}$ in $\mathbb{C} \setminus \{0\}$ with $\lambda_i \xrightarrow{i \to \infty} 0$ (if I infinite)
such that:

$$X = \operatorname{Ker}(T) \oplus^{\perp} \overline{\operatorname{Span}(e_i \mid i \in I)}$$

orthogonal sum: $X = U \oplus^{\perp} V$ means:
for each $x \in X$ there is $u \in U$, $v \in V$:
- $x = u + v$
- $u \perp v$

unique!

and $$Tx = \sum_{k \in I} \lambda_k e_k \langle e_k, x \rangle \quad \text{for } x \in X$$

eigenvector to λ_k

eigenvalue

and $$\|T\| = \sup_{k \in I} |\lambda_k| .$$