Functional analysis - part 5

Example: $X = (0,3)$ with $d(x,y) = |x-y|$

$(0,3)$ is closed:

- complement $\emptyset$ is open
- each convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq (0,3)$ (with limit $\tilde{x} \in X$) satisfies $\tilde{x} \in (0,3)$

What is about the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$?

- sequence in X
- $d(x_n, x_m) \xrightarrow{n,m\to\infty} 0$
- it does <u>not</u> converge $\Rightarrow$ (X,d) is not complete

Definition: Let (X,d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called <u>Cauchy sequence</u> if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n,m \geq N : d(x_n, x_m) < \varepsilon$.

(X,d) is called <u>complete</u> if all Cauchy sequences converge.

Example: (a) $X = [0,3]$ with $d(x,y) = |x-y|$ is complete.

(b) $X = (0,3)$ with $d(x,y) = \begin{cases} 1 & , x \neq y \\ 0 & , x = y \end{cases}$ is complete.

Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Take $\varepsilon = \frac{1}{2}$. Then there is an $N\in\mathbb{N}$ such that for all $n,m \geq N$, we have $\underbrace{d(x_n, x_m)}_{=0} < \frac{1}{2}$. Hence $x_n = x_m$.