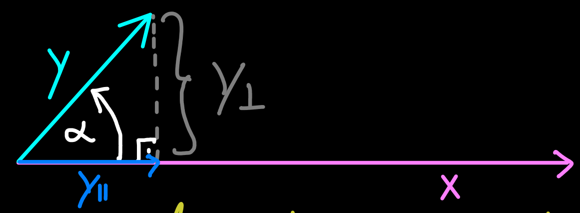
Functional analysis - part 10

Cauchy-Schwarz inequality: Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and $\|x\| := \sqrt{\langle x, x \rangle}$. Then for all $x, y \in X$:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$



and $|\langle x, y\rangle| = \|x\| \cdot \|y\| \iff x, y$ linearly dependent

Proof: 1st case: $x = 0$: $|\langle x, y\rangle| = 0 = \|x\| \cdot \|y\|$ ✓

2nd case $x \neq 0$: $\hat{x} := \frac{x}{\|x\|}$, $\quad y_\parallel := \langle \hat{x}, y\rangle \hat{x}$, $\quad y_\perp := y - y_\parallel$

$$0 \leq \|y_\perp\|^2 = \|y - y_\parallel\|^2 = \|y - \langle \hat{x}, y\rangle \hat{x}\|^2 = \langle y - \langle \hat{x}, y\rangle \hat{x}, y - \langle \hat{x}, y\rangle \hat{x}\rangle$$

$$= \langle y - \langle \hat{x}, y\rangle \hat{x}, y\rangle - \langle y - \langle \hat{x}, y\rangle \hat{x}, \langle \hat{x}, y\rangle \hat{x}\rangle$$

$$= \langle y, y\rangle - \langle \langle \hat{x}, y\rangle \hat{x}, y\rangle - \langle y, \langle \hat{x}, y\rangle \hat{x}\rangle + \langle \langle \hat{x}, y\rangle \hat{x}, \langle \hat{x}, y\rangle \hat{x}\rangle$$

$$= \|y\|^2 - \left(\langle \langle \hat{x}, y\rangle \hat{x}, y\rangle + \overline{\langle \langle \hat{x}, y\rangle \hat{x}, y\rangle}\right) + \|\langle \hat{x}, y\rangle \hat{x}\|^2$$

$$= \|y\|^2 - \left(2 \cdot \mathrm{Re}(\underbrace{\langle \langle \hat{x}, y\rangle \hat{x}, y\rangle}_{\overline{\langle \hat{x}, y\rangle} \langle \hat{x}, y\rangle = |\langle \hat{x}, y\rangle|^2})\right) + |\langle \hat{x}, y\rangle|^2 \cdot \underbrace{\|\hat{x}\|^2}_{=1}$$

$$= \|y\|^2 - 2|\langle \hat{x}, y\rangle|^2 + |\langle \hat{x}, y\rangle|^2 = \|y\|^2 - |\langle \hat{x}, y\rangle|^2$$

$$\Rightarrow \quad \|y\|^2 \geq |\langle \hat{x}, y\rangle|^2 = \left|\left\langle \tfrac{x}{\|x\|}, y\right\rangle\right|^2 = \frac{1}{\|x\|^2} |\langle x, y\rangle|^2$$

$$\Rightarrow \quad \|x\| \cdot \|y\| \geq |\langle x, y\rangle|$$

Δ-inequality for $\|\cdot\|$:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle) + \|y\|^2$$

$$\leq \|x\|^2 + 2\,|\langle x, y\rangle| + \|y\|^2$$

$$\overset{\text{Cauchy Schwarz}}{\leq} \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$