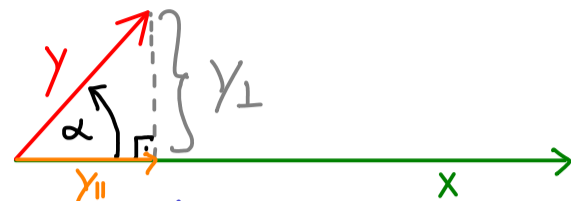


## Functional analysis - part 10

Cauchy-Schwarz inequality: Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space and  $\|x\| := \sqrt{\langle x, x \rangle}$ . Then for all  $x, y \in X$ :

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

and  $|\langle x, y \rangle| = \|x\| \cdot \|y\| \iff x, y$  linearly dependent



Proof: 1st case:  $x = 0$ :  $|\langle x, y \rangle| = 0 = \|x\| \cdot \|y\|$  ✓

2nd case  $x \neq 0$ :  $\hat{x} := \frac{x}{\|x\|}$ ,  $y_{\parallel} := \langle \hat{x}, y \rangle \hat{x}$ ,  $y_{\perp} := y - y_{\parallel}$

$$\begin{aligned} 0 \leq \|y_{\perp}\|^2 &= \|y - y_{\parallel}\|^2 = \|y - \langle \hat{x}, y \rangle \hat{x}\|^2 = \langle y - \langle \hat{x}, y \rangle \hat{x}, y - \langle \hat{x}, y \rangle \hat{x} \rangle \\ &= \langle y - \langle \hat{x}, y \rangle \hat{x}, y \rangle - \langle y - \langle \hat{x}, y \rangle \hat{x}, \langle \hat{x}, y \rangle \hat{x} \rangle \\ &= \langle y, y \rangle - \langle \langle \hat{x}, y \rangle \hat{x}, y \rangle - \langle y, \langle \hat{x}, y \rangle \hat{x} \rangle + \langle \langle \hat{x}, y \rangle \hat{x}, \langle \hat{x}, y \rangle \hat{x} \rangle \\ &= \|y\|^2 - \left( \langle \langle \hat{x}, y \rangle \hat{x}, y \rangle + \overline{\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle} \right) + \|\langle \hat{x}, y \rangle \hat{x}\|^2 \\ &= \|y\|^2 - \left( 2 \cdot \text{Re}(\langle \langle \hat{x}, y \rangle \hat{x}, y \rangle) \right) + |\langle \hat{x}, y \rangle|^2 \cdot \underbrace{\|\hat{x}\|^2}_{=1} \\ &= \|y\|^2 - 2 \underbrace{|\langle \hat{x}, y \rangle|}_{\substack{= \\ \langle \hat{x}, y \rangle \langle \hat{x}, y \rangle}} |\langle \hat{x}, y \rangle| + |\langle \hat{x}, y \rangle|^2 = \underbrace{\|y\|^2 - |\langle \hat{x}, y \rangle|^2}_{\text{Cauchy-Schwarz}} \end{aligned}$$

$$\Rightarrow \|y\|^2 \geq |\langle \hat{x}, y \rangle|^2 = \left| \left\langle \frac{x}{\|x\|}, y \right\rangle \right|^2 = \frac{1}{\|x\|^2} |\langle x, y \rangle|^2$$

$$\Rightarrow \|x\| \cdot \|y\| \geq |\langle x, y \rangle|$$

$\Delta$ -inequality for  $\|\cdot\|$ :

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle = \|x\|^2 + 2 \text{Re}(\langle x, y \rangle) + \|y\|^2 \\ &\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2 \\ &\stackrel{\text{Cauchy-Schwarz}}{\leq} \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \end{aligned}$$