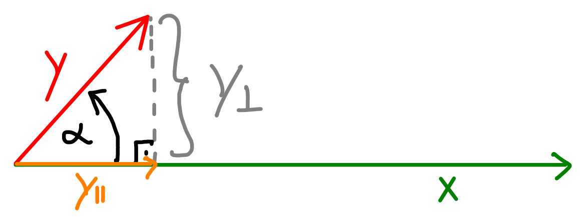
Functional analysis - part 11

Orthogonality: Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space.



(a) $x, y \in X$ are called orthogonal if $\langle x, y \rangle = 0$. Write $x \perp y$.

(b) For $U, V \subseteq X$, write $U \perp V$ if $x \perp y$ for all $x \in U$, $y \in V$.

(c) For $U \subseteq X$, the orthogonal complement of U is

$$\underbrace{U^\perp}_{\text{is always a subspace in } X} := \{ x \in X \mid \langle x, u \rangle = 0 \text{ for all } u \in U \}$$

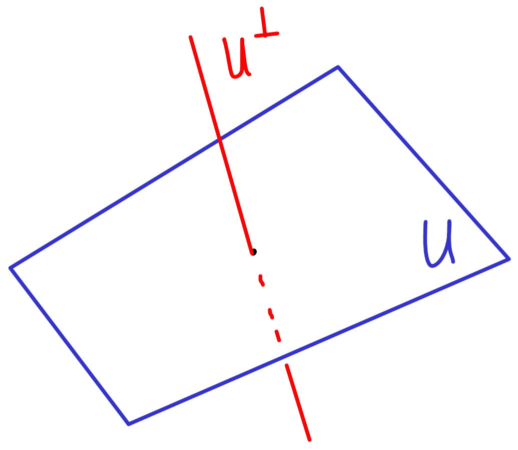
Remark: (1) $\{0\}^\perp = X$, $\quad X^\perp = \{0\}$

(2) $U \subseteq V \implies U^\perp \supseteq V^\perp$

Proof: $x \in V^\perp \implies \langle x, v \rangle = 0$ for all $v \in V$

$\overset{U \subseteq V}{\Longrightarrow} \langle x, u \rangle = 0$ for all $u \in U \implies x \in U^\perp$

(3) If $x \perp y$, then $\|x+y\|^2_{\langle \cdot,\cdot \rangle} = \|x\|^2_{\langle \cdot,\cdot \rangle} + \|y\|^2_{\langle \cdot,\cdot \rangle}$ (Pythagorean theorem)



$U^\perp$ is always closed