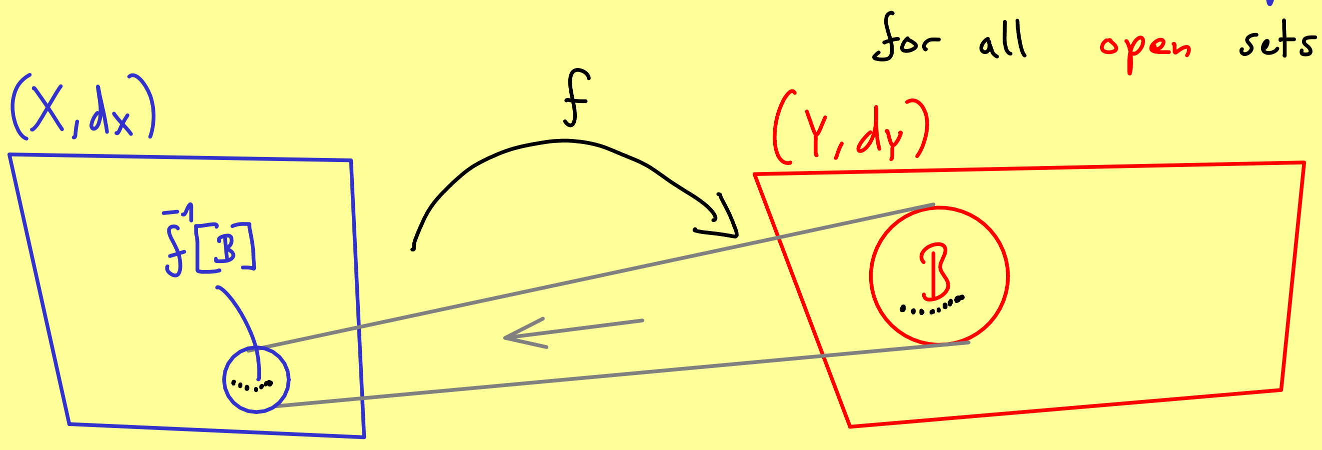
The Bright Side of Mathematics

Functional analysis – part 12

Continuity for metric spaces: (X, d_X), (Y, d_Y) two metric spaces.

A map $f: X \to Y$ is called:

- continuous if $f^{-1}[B]$ is open (in X) for all open sets $B \subseteq Y$.



- sequentially continuous if for all $\tilde{x} \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} \tilde{x}$ holds $f(x_n) \xrightarrow{n\to\infty} f(\tilde{x})$.

Fact: For metric spaces, continuous and sequentially continuous are equivalent.

Examples: (a) (X, d_X) discrete metric space, (Y, d_Y) any metric space

$\Rightarrow$ all $f: X \to Y$ are continuous

(b) $(X, d_X), (Y, d_Y)$ metric spaces, $y_0 \in Y$ fixed.

$\Rightarrow$ $f: X \to Y$, $x \mapsto y_0$ is always continuous.

(c) $(X, \|\cdot\|)$ normed space, $Y = \mathbb{R}$ with standard metric

$\Rightarrow$ $f: X \to \mathbb{R}$, $x \mapsto \|x\|$ is continuous

Proof: Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$f(x_n) = \|x_n\| = \|x_n - \tilde{x} + \tilde{x}\| \overset{\Delta\text{-inequ.}}{\leq} \|x_n - \tilde{x}\| + \|\tilde{x}\| = \underbrace{d(x_n, \tilde{x})}_{\xrightarrow{n\to\infty} 0} + f(\tilde{x})$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) \leq f(\tilde{x})$$

$$f(\tilde{x}) = \|\tilde{x}\| = \|\tilde{x} - x_n + x_n\| \overset{\Delta\text{-inequ.}}{\leq} \|\tilde{x} - x_n\| + \|x_n\| = d(\tilde{x}, x_n) + f(x_n)$$

$$\Rightarrow f(\tilde{x}) \leq \lim_{n\to\infty} f(x_n) \qquad \square$$

(d) $(X, \langle\cdot,\cdot\rangle)$ inner product space, $Y = \mathbb{C}$ with the standard metric, $x_0 \in X$ fixed.

$\Rightarrow$ $f: X \to \mathbb{C}$, $x \mapsto \langle x_0, x\rangle$ is continuous

Proof: Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ sequence with limit $\tilde{x} \in X$. Then:

$$|f(x_n) - f(\tilde{x})| = |\langle x_0, x_n\rangle - \langle x_0, \tilde{x}\rangle| = |\langle x_0, x_n - \tilde{x}\rangle|$$

$$\overset{\text{C.S.}}{\leq} \|x_0\| \cdot \underbrace{\|x_n - \tilde{x}\|}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to\infty} 0$$

Analogously, $g: X \to \mathbb{C}$, $x \mapsto \langle x, x_0\rangle$ is continuous.

Claim: $(X, \langle\cdot,\cdot\rangle)$ inner product space, $U \subseteq X$. Then $U^\perp$ is closed.

Proof: Let $(x_n)_{n\in\mathbb{N}} \subseteq U^\perp$ with limit $\tilde{x} \in X$.

$\Rightarrow$ $\langle x_n, u\rangle = 0$ for all $u \in U$

$\Rightarrow$ $\lim_{n\to\infty} \langle x_n, u\rangle = 0$ for all $u \in U$

$\Rightarrow$ $\langle \tilde{x}, u\rangle = 0$ for all $u \in U$ $\Rightarrow$ $\tilde{x} \in U^\perp$ $\qquad \square$