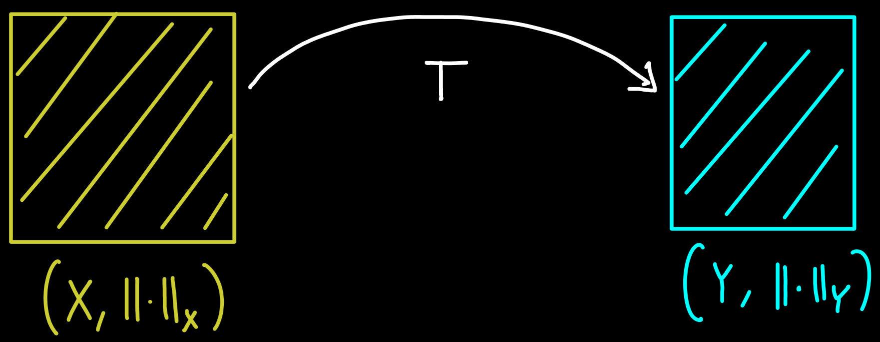
Functional analysis – part 13

Operator:



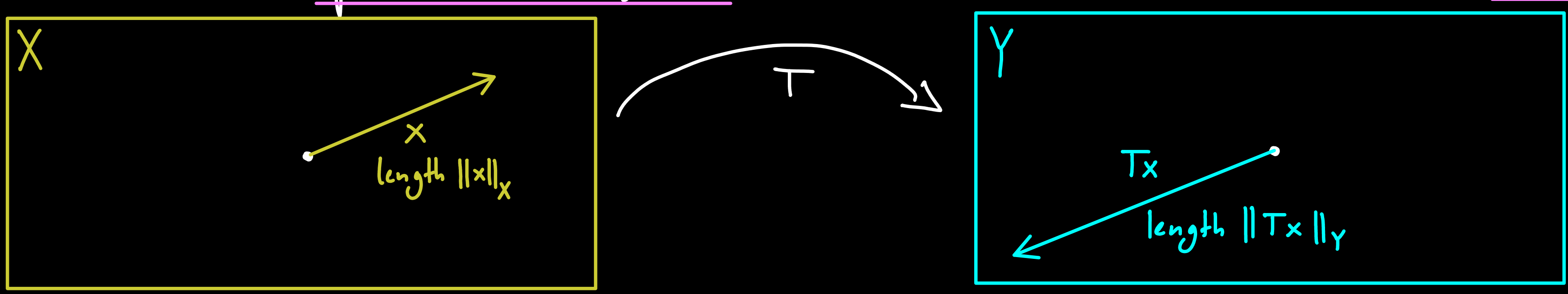
$T: X \longrightarrow Y$:
- linear (conserves the algebraic structure)
- continuous (bounded) (conserves the topological structure)

<u>Definition:</u> $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces, $T: X \longrightarrow Y$ linear

$\begin{cases} T(x+\tilde{x}) = Tx + T\tilde{x} \\ T(\lambda x) = \lambda Tx \end{cases}$ for all $x, \tilde{x} \in X$, $\lambda \in \mathbb{F}$

$$\|T\| = \|T\|_{X \to Y} := \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X,\ x \neq 0 \right\}$$

is called the <u>operator norm of T</u>. If $\|T\| < \infty$, T is called <u>bounded</u>.



Proposition: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces, $T: X \longrightarrow Y$ linear.
Then the following claims are equivalent:

(a) T is continuous.
(b) T is continuous at $x = 0$.
(c) T is bounded.

Proof: (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c): $(*)$ For all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} 0$, we have $Tx_n \xrightarrow{n\to\infty} 0$.

Claim: $(*) \Rightarrow$ [There is a $\delta > 0$ such that $\|Tx\|_Y < 1$ for all $x \in X$ with $\|x\|_X < \delta$] $(\ast)$

Proof of the claim: $\neg(\ast) \Rightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in X$ with $\|x_n\|_X < \frac{1}{n}$ and $\|Tx_n\|_Y \geq 1$ $\Rightarrow \neg(*)$

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|Tx\|_Y \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}}{\|x\|_X \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}} = \frac{\| T(\frac{\delta}{2} \frac{x}{\|x\|_X}) \|_Y}{\underbrace{\| \frac{\delta}{2} \frac{x}{\|x\|_X} \|_X}_{= \frac{\delta}{2}}} \leq \frac{2}{\delta}$$

$$\Rightarrow \|T\| = \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \;\middle|\; x \in X,\ x \neq 0 \right\} \leq \frac{2}{\delta} < \infty$$

(c) $\Rightarrow$ (a): Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be convergent with limit $\tilde{x} \in X$. Then

$$\| Tx_n - T\tilde{x} \|_Y = \| T(x_n - \tilde{x}) \|_Y \leq \|T\| \cdot \|x_n - \tilde{x}\|_X \xrightarrow{n\to\infty} 0 \qquad \square$$