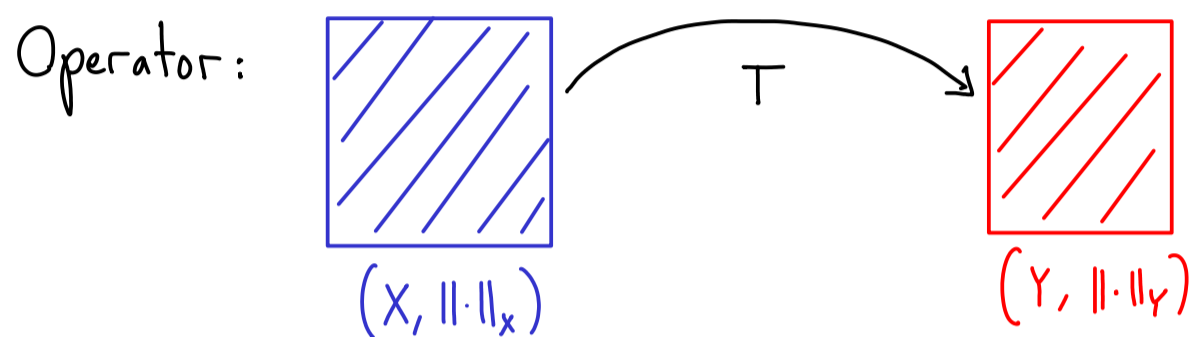


Functional analysis - part 13



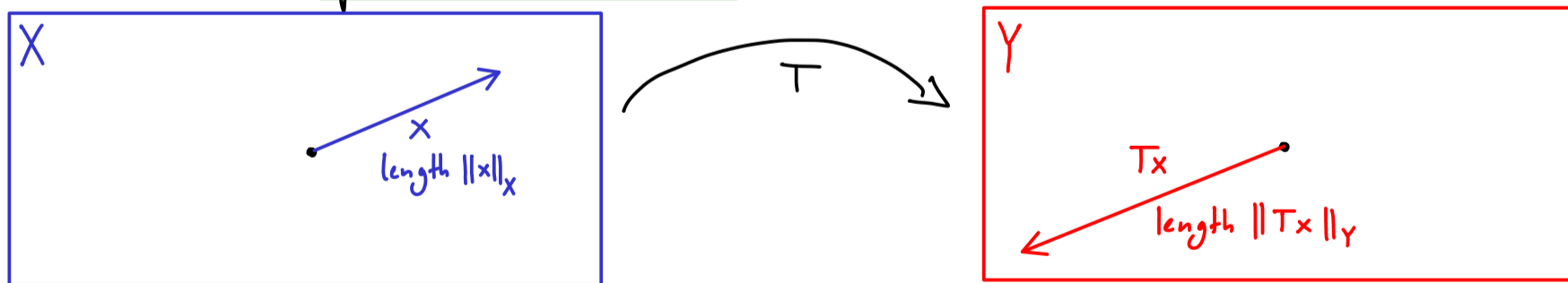
- $T: X \rightarrow Y$:
- linear (conserves the algebraic structure)
 - continuous (bounded) (conserves the topological structure)

Definition: $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ two normed spaces, $T: X \rightarrow Y$ linear

$\|T\| = \|T\|_{X \rightarrow Y} := \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \mid x \in X, x \neq 0 \right\}$

$\left\{ \begin{array}{l} T(x+\tilde{x}) = Tx + T\tilde{x} \\ T(\lambda x) = \lambda Tx \end{array} \right.$ for all $x, \tilde{x} \in X, \lambda \in \mathbb{F}$

is called the operator norm of T. If $\|T\| < \infty$, T is called bounded.



Proposition: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces, $T: X \rightarrow Y$ linear.

Then the following claims are equivalent:

- (a) T is continuous.
- (b) T is continuous at $x=0$.
- (c) T is bounded.

Proof: (a) \Rightarrow (b) \checkmark

(b) \Rightarrow (c): (*) For all sequences $(x_n)_{n \in \mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n \rightarrow \infty} 0$, we have $Tx_n \xrightarrow{n \rightarrow \infty} 0$.

Claim: (*) \Rightarrow [There is a $\delta > 0$ such that $\|Tx\|_Y < 1$
for all $x \in X$ with $\|x\|_X < \delta$] (*)

Proof of the claim: $\neg(*) \Rightarrow$ For all $n \in \mathbb{N}$, we find $x_n \in X$ with $\|x_n\|_X < \frac{1}{n}$
and $\|Tx_n\|_Y \geq 1 \Rightarrow \neg(*)$

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|Tx\|_Y \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}}{\|x\|_X \cdot \frac{\delta}{2} \cdot \frac{1}{\|x\|_X}} = \frac{\|T(\frac{\delta}{2} \frac{x}{\|x\|_X})\|_Y}{\|\frac{\delta}{2} \frac{x}{\|x\|_X}\|_X} \leq \frac{2}{\delta}$$

$$\Rightarrow \|T\| = \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \mid x \in X, x \neq 0 \right\} \leq \frac{2}{\delta} < \infty$$

(c) \Rightarrow (a): Let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be convergent with limit $\tilde{x} \in X$. Then

$$\|Tx_n - T\tilde{x}\|_Y = \|T(x_n - \tilde{x})\|_Y \leq \|T\| \cdot \|x_n - \tilde{x}\|_X \xrightarrow{n \rightarrow \infty} 0 \quad \square$$