

Functional analysis - part 14

Example: $X = (C([0,1], \mathbb{F}), \|\cdot\|_\infty)$, $Y = (\mathbb{F}, |\cdot|)$

For $g \in X$ with $g(t) \neq 0$ for all $t \in [0,1]$, define

$$T_g: X \rightarrow Y \quad \text{by} \quad T_g(f) := \int_0^1 g(t) \cdot f(t) dt$$

What is $\|T_g\|$?

$$\begin{aligned} \|T_g\| &= \sup \left\{ \frac{|T_g(f)|}{\|f\|_\infty} \mid f \in X, f \neq 0 \right\} \\ &= \sup \left\{ |T_g(f)| \mid f \in X, \|f\|_\infty = 1 \right\} \\ &= \sup \left\{ \left| \int_0^1 g(t) \cdot f(t) dt \right| \mid f \in X, \|f\|_\infty = 1 \right\} \\ &\leq \int_0^1 |g(t)| \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty = 1} dt \\ &\leq \int_0^1 |g(t)| dt < \infty \end{aligned}$$

Check the other inequality: $h(t) := \frac{\overline{g(t)}}{|g(t)|}$ with $\|h\|_\infty = 1$

$$\|T_g\| \geq |T_g(h)| = \left| \int_0^1 g(t) \frac{\overline{g(t)}}{|g(t)|} dt \right| = \int_0^1 \frac{|g(t)|^2}{|g(t)|} dt = \int_0^1 |g(t)| dt$$